THE MOZAMBICAN STUDENTS' UNDERSTANDING OF THE CONCEPT LIMIT OF A FUNCTION: A CASE STUDY

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DECLARATION

I declare that this research report is my own, unaided work. It is being submitted for the Degree of Master of Science in Education in the University of Witwatersrand, Johannesburg. It has not been submitted in any other University.

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(Signature of candidate)

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Abstract

The limit concept, a basic concept for Calculus courses has been reported as a difficult concept for the students around the world. As a teacher at a secondary school and as a lecturer at the Eduardo Mondlane University I perceived the students' difficulties when encountering this concept. The present study scrutinizes the students' understanding of the concept of limit of a function. I borrowed the concepts *concept image* and *concept definition* from Tall and Vinner (1981) to establish the students' mental images and definitions of the limit concept. Since mental images can only come into being within a representation, I concerned myself with how the students worked within the numerical, algebraic and graphical representations and how they switched from one representation to another, drawing on Douady's (1986) work.

Two classes of two schools, one in Maputo and other in Quelimane were chosen for the present study. A task was administered and nine students, four from Maputo and five from Quelimane, were interviewed with the goal of deepening the students' thinking when tackling the limit concept.

The results of the research showed that students' concept definitions of a limit of a function might be split into two broad categories. The first category encompassed the definitions that suggested a dynamic nature which were classified as *approach* and *continuity.* Alternatively, the students' definitions indicated a static concept definition, using the words interval, neighborhood or the formal definition with the $\varepsilon - \delta$ symbols and were included in the categories *notation/algebraic calculation*, *boundary* and *formal definition*. Similarly, the students' images were categorized as dynamic or static in nature. The dynamic images included the *asymptotic* and the *motion picture.* Adding to these two categories three static concept images were classified: the *barrier*, the *value correspondence* and the *procedural.* Moreover, I established the relationship between the different representation systems and students' concept images used in the solution process. In addition, I ascertained students' connections between images associated to the context and those generated from the mathematics in a daily life problem.

Dedication

I would like to dedicate my thesis to my parents Abiatar and Alfina, specially my mother who sacrificed to give me education. To my husband who dedicated this time, effort and patience to ensure that I completed my research report. My three children, Cíntia, Sumila and Celma, are still very young and they have missed out, in terms of my attention, time to play, assistance in school stuff, and being together as a family. Without Jose Pedro's effort, this could have become a problem. I thank them for their love and for giving me the opportunity of doing my Degree.

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CHAPTER 1

INTRODUCTION

1.1 STATEMENT AND AIMS OF THE PROBLEM

"Define the function $f(x)$ by letting $f(x)$ be the distance from a certain train to the station at time x, where x is measured in hours after 12:00 noon on March 1, 1989. At exactly 2:00 p.m. on that day, the train arrives and comes to a complete stop at the station. Discuss the limit of $f(x)$ as $x \to 2$" (Williams, 1991, p. 224). Williams (1991) used this question in his study to alter the students' spontaneous models of limits of functions. One of these models was whether a function could reach its limit. A student claimed in an interview,

> "I thought about the train example (laughter). This is going to get really philosophical, maybe, but like the train comes to a stop at like a certain time, right but it's ... what is stopped? I mean you can question the words "no movement". Does anything ever stop moving? I mean, is there such a thing as no motion? And you can say, slows down enough for passengers to come in and get out, but what is exactly stopping, unless there is like, I supposed at a certain temperature level, there's absolutely no motion. But when you take it to that extreme, there's absolutely no motion, otherwise, how do you define stopping?" (Williams, 1991, p. 227)

The student tries to deny her experience to be consistent with her concept image of a limit as something that is not reached. Along with this concept image the student held other images that hindered her/his understanding of the limit concept. As a teacher at a secondary school and later a lecturer at a University I too have been faced with students struggling with this concept. Thus, the purpose of this study is to find out how Mozambican students deal with the concept of a limit of a function when they come from secondary school, where they are taught the concept in Grade 12. To make sense of the students' concepts I will use the ideas of *concept image* and *concept definition* as described by Tal and Vinner (1981). Besides that I want to access how the different representations - theoretical (definition), graphical, numerical and algebraic - influence their understanding of the concept of a limit of a function.

1.2 RESEARCH QUESTIONS

The questions I intend to answer in my research report are the following:

1. Do the students conceive of the limit as a static number or as a dynamic process?
2. What concept image and concept definition of the limit of a function do the Mozambican Grade 12 students hold?
3. How do the Mozambican Grade 12 students understand each representation and how do they relate the different representations: theoretical (definition), graphical, numerical and algebraic?

However, the present study was not concerned with the formal definition using $\epsilon - \partial$ symbols. The reason for this is that this definition is very abstract and the Mozambican students do not understand it and Mozambican teachers rarely teach this definition (Huillet and Mutemba, 1999). Recently researchers have stated that the formal definition of limit of a function is viewed as difficult for students because of the use of quantifiers ($\forall$ **and** $\exists$)(Tall and Vinner, 1981) and the use of the $\epsilon - \partial$ symbols (Vinner, 1991; Huillet and Mutemba, 1999; Espinoza and Azcárete, 1995).

1.3 DEFINITION OF TERMS

In my report I will use some scientific terms such as concept image, concept definition and so on. In this section I am going to explain in which sense I use them.

A *concept image* is the total cognitive structure that is associated with a concept, which includes all the mental pictures and associated properties and processes. (Vinner, 1991). The concept image of the limit of a function can be associated not only with the definition but also with the algebraic techniques used to calculate limits, indeterminate forms, the graphical representation of asymptotes, and the continuity of a function or the sequence of x values approaching a point. One uses *representation systems* to express his (her) mental images. Douady (1986) defines a *representation* as a set of mathematical objects and relations between them, formulations sometimes contradictory and mental images associated to these objects or images. In my report I considered the algebraic, the numerical and the graphical representations.

A limit of a function can be seen in two ways: limit as a *dynamic* process and as a *static* number. The limit is viewed as a *dynamic process* when one thinks of it as movement. That is, the value of the function is approaching a number more and more closely.

For example, $0,\dot{9} = 1$ because the sequence of values approaches 1. Tall (1992) suggests phrases such as "u_n *tends to l*", "u_n *approaches to l*", "*the distance from* u_n *to l becomes small*", etc indicate a motion or a *dynamic* process. Alternatively, the motion idea may not be considered. One can consider, a, the limit of a function because the values of the function are in an interval near a, or are grouped around the number a. Tall (1992) uses the phrases "u_n *is as close as you like to l*", "*the* u_n *are in an interval near l*" to show the limit as a *static object*.

Concept definition is the form of words used to specify a concept (Tall &Vinner, 1981, p.152). The limit concept may be defined by means of the formal definition with the $\varepsilon - \delta$ symbol or by means of a dynamic definition.

When one defines a limit dynamically expressions such as 'tends to', 'approaches to', 'limit' and 'converges' are employed. These words are also used in daily life. Therefore, students already hold images of these words based on their experiences outside the classroom. These words have a certain sense for the students different from that hold by the mathematics community. This daily life sense is called *ordinary sense* in the present report. Cornu (1991) names these images held by the students before formal teaching, *spontaneous conceptions*. According to him, these conceptions do not disappear with formal teaching, rather they mix with the new conceptions and form the individual's concept images.

1.4 THE RATIONALE OF THE STUDY

The concept of "limits of functions' is fundamental for "the Calculus" at a University. In Mozambique this concept is first introduced to students in Grade 12. The concept can be represented in various ways, for example the numerical representation, the graphical representation and the algebraic representation. From my experience as a lecturer at Eduardo Mondlane University, many students struggle to deal with the graphical and the numerical representations. Furthermore they have forgotten the definition, perhaps because they have not understood it. They are able to manage the calculations of limits of functions (algebraic representation) without many problems, but they do not see the link between the different representations. For instance, when they have a calculus result, they do not know what it means in the graphical setting. In addition to this, they are not used to managing the numerical representation. They think that the problem is solved with the substitution of the x value and successive algebraic calculations. I decided to investigate which *concept definition* the students hold and identify the possible explanations for misunderstandings in the students' *concept image*. I hope that the issues raised in this study help Mozambican policy makers and teachers to improve the methods of approaching the limit concept. Important features may be taken into account in the teaching process. For instance, teachers may discuss with the students the meaning of the terms used in the limiting process so that the spontaneous conceptions do not influence their individual concept images negatively. In addition, highlighting interaction between the different representations of a limit and dealing with problems that represent situations where the limit concept is applied may enhance the evolution of the concept formation.

1.5 OUTLINE OF THE RESEARCH

With this study I intend to gain access to the Mozambican students' conceptions of the limit concept near the end of secondary school. I based my analysis on Tall and Vinner's (1981) concept definition and concept image. Since the students' concept image and concept definitions need a device to come into being, I used, in addition, Douady's (1986) concept of representation. Therefore the students' concept image and concept definitions were analysed in the light of their proficiency in working within the various representations. In Chapter 2 I review some preceding studies about the limit concept in order to gain insight into student pitfalls reported by other researchers. In addition I looked for other literature where *concept definition*, *concept image* and *representation* were applied. I concerned myself with existing literature about concepts related to my study such as the spontaneous conceptions (Cornu, 1991). In Chapter 3 I present an overview of the sample, the instruments I used to gather the data and how the data was analysed. Chapter 4 is the essence of this study. In this chapter I amassed the data from the students' answers from the task and the data from the transcription of the interviews, placing them in the concept definitions and concept images categories drawn from Miles' (1984) work. In addition I looked for the students' proficiency switching from one representation to another. Finally, Chapter 5 is concerned with the summary of my findings as they relate to the research questions. Moreover, I analysed the interplay between the students' concept definition and concept images. The relationship between concept images and the representation systems was also considered in Chapter 5. Furthermore, I addressed some implications, limitations of the study and raised some issues for further studies.

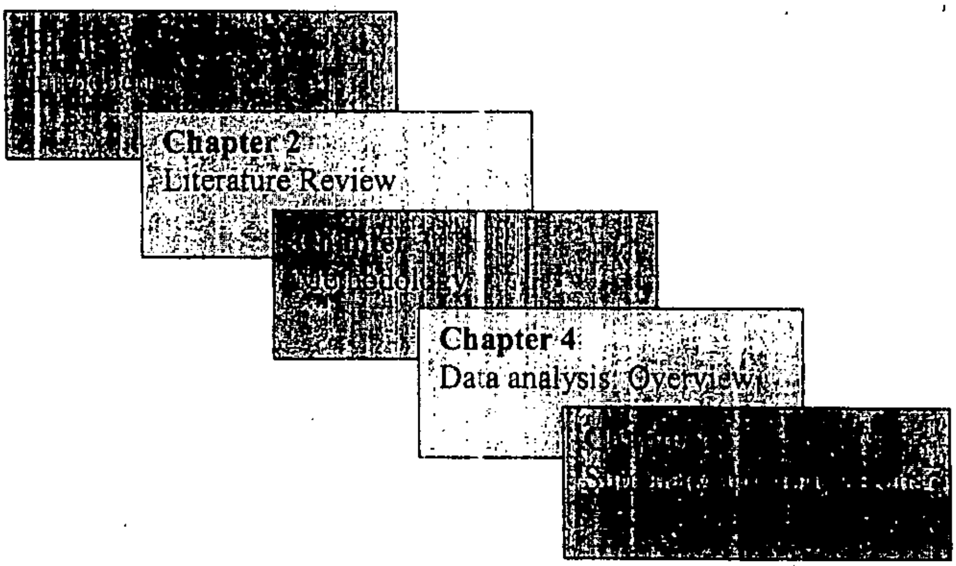


Figure 1.1 Chapter overview

CHAPTER 2

LITERATURE REVIEW

2.1 INTRODUCTION

The concept of "limit of a function" is the basis for the subject "Calculus". The Mozambican first year students encounter many difficulties when they are faced with this concept, which is taught in Grade 12 in Mozambique. Researchers have pointed to the difficulties students of various countries experience with this concept. Researchers have paid attention to students' misconceptions in the learning of limits of functions in recent years from different perspectives. Cornu (1991) and Sierpinska (1987) focus their attention on "epistemological obstacles", Tall and Vinner (1981) emphasize the importance of the "concept image", Cornu (1991) centres his attention on the "spontaneous models" and still other researchers were concerned with "didactical engineering" (Robinet, 1983 Trouche 1996).

Obstacles in constructing the limit concept

Cornu (1991) raised a number of obstacles students face when constructing the concept of limit of a function. He adds that these obstacles have already appeared in the historical development of the concept and points out four major obstacles that emerged in the history of the development of the limit concept: the metaphysical aspect of the notion of limit of a function, the notion of infinitely large and infinitely small, the failure to link geometry with number and the conception that the limit is never attained.

Firstly, students view the notion of *limit* as difficult because of its metaphysical features, that is, its unusual and abstract procedures. Cornu (1991) ascribes the notions of infinity and limit as relevant to the metaphysical aspects. The procedures of algebra and arithmetic that have been used up to now are no longer helpful to solve limits because the students have to deal with the concept of infinity. For example, ST, a student from Williams's (1991) study expressing her metaphysical belief claimed, "*And I thought about all the definitions that we dealt with, and I think they're all right - they're all correct in a way and they're all incorrect in a way because they can only apply to a certain number of functions, while others apply to other functions, but it's like talking about infinity or God, you know. Our mind is only so limited that you don't know the real answer, but part of it*". In addition, the concepts infinitely large and infinitely small are difficult to understand. Students do not imagine a positive number that is so small it is almost zero, or a big number, bigger than any other. The introduction of the symbol ε expressing a positive quantity that is close to zero but not zero leads the students to have a dynamic view thinking of an unfinished process.

Secondly, the most common geometric problem linked with limits is the calculation of the area of a circle, increasing the number of the sides of an inscribed regular polygon. The students have difficulties in understanding the passage to limit when the number of sides increases.

Thirdly, one of the student's difficulties lies on understanding that the function may assume its limit value. This debate has lasted through the history of the concept. Cornu (1991) cited D'Alembert who said, "The limit never coincides or never becomes equal to a quantity of which it is the limit, but is always approaching and can differ by as small a quantity one desires" (p. 162). The following protocols, showing the students' belief about the limit not being reached, were extracted from Williams's (1991) study. A student, JKLM claimed, "*Well I would say that the function reaches its limit. But not when you're trying to approach that number. But like after that, if you plug in the right number it can reach its limit*" (Williams, 1991, p.226). In the same way, DT stated, "*As far as the function goes, you know, if you don't do the limit or anything on it, you can plug in that 1 and you're going to get the* *value. You're going to get the exact value. But that would be a totally different separate problem, I mean nothing to do with limits*". For the students the process of plugging in points, and therefore reaching the limit has nothing to do with the evaluation of a limit, because they conceive of the limit as something that is not reached. Likewise, Szydlik (2000) pointed out that the students may think that the values of the function get closer to the limit but never equal it, because they imagine the existence of an infinitesimal distance between the limit and the values of the function. The previous examples demonstrated that the students separate the function in a limiting context from the function in a more general situation. Assigning the students' belief to their practice in the classroom, Tall (1991) argued that the sequence $u_n = \frac{1}{n}$ is in most cases given to the students as an example of a convergent sequence. This misleads the students to think that in all cases the terms of the sequence tend but never assume the value of the limit, there is always a slight difference between the limit and any term of the sequence. The introduction of counter-examples in the teaching process would contradict this conception. Moreover, the limit is viewed as a computational method, a process of coming near to a value, rather than a result, the value of the limit itself. Many studies have reported students' difficulties in understanding that $0,99 = 1$ (Schwarzenberger and Tall, 1978; Sierspinska, 1987). One of the reasons evoked for the students' conception of 0,99.... being less then 1 is the notion of 'infinitesimal' that is regarded as 'infinitely close but not equal' (Schwarzenberger and Tall, 1978). According to Artigue (1996) the first term of the equality represents the limit as a process and the second as an object. She also said that to understand this equality it is essential to distinguish the infinite process described by 0,99.... and the number resulting from this process. These twofold features of the concept limit, as a process and as an object simultaneously, denominated a procept, are ascribed to Gray and Tall (in Monaghan et. al., 1994). These obstacles may be a hindrance for a student's formation of the limit conception. The difficulties presented by the students in the study were in most cases assigned to misconceptions related to the mentioned obstacles. Actually, *concept images* of the limit concept they developed were influenced, to some extent, by the way they conceived of infinity or the limit as a process or as an object. Students' *concept images* were used in the study to ascertain their understanding of the limit concept. The state of a student's *concept image* may be established by the way s/he works within a representation and his (her) ability to switch from one representation to another. Therefore I also used the concept of representation as framework of my study.

2.2 CONCEPT DEFINITION AND CONCEPT IMAGES

In the process of learning students are no longer considered empty vessels that only receive knowledge from the teacher. Contrary, they bring to the classroom some notions of a concept before it is taught For instance, the students use in their daily life the word *limit*. Consequently, when they first encounter the concept of limit at school they already have some conceptions of this word. These conceptions are ideas, intuitions, images, and knowledge, which are acquired through experience. Cornu (1991) referred to these as *spontaneous conceptions* linked to the limit concept. He goes on to say that these ideas that developed prior to formal teaching do not disappear after teaching. Instead, the students' old concepts are linked to the new one, modified and adapted to form a revised conception. Some researchers (Cornu, 1980; Tall and Schwerzenberger, 1978; Tall and Vinner 1981) considered the everyday meanings of terms also associated with mathematical concepts a possible hindrance for the learning of the mathematical concepts. Actually words such as *limit*, *approaches*, *tends to* and *converges* have a different meaning in daily life and can interfere with the formation of an appropriate conception of the limit concept. Monaghan (1991) using the same mathematical words concluded that these words interfere with the students' understanding of the concept of limit of a function. On the other hand, Pimm (1981) stressed that confusion may emerge from the difference between teachers' and students' meaning of a concept. This difference is credited to the difference between ordinary language and mathematical language. In Sfard et.al. (1998), Nesher suggested that "talking mathematically" to students helps the teacher evaluate what they have understood about a concept, but it cannot help the students understand the concept. She said, "Students will be evaluated according to their ability to talk mathematically in using this term, and not by the way they talk about it in natural language". Therefore, teachers must be aware of the interference of the spontaneous conceptions of the terms used to describe the limit concept and consequently "talk mathematically" about them to students so that the *concept images* they hold do not differ too extensively from those conceived by mathematicians. According to Vinner (1991, p. 68)

> The concept image is something non-verbal associated in our mind with the concept name. It can be a visual representation of the concept in case the concept has visual representations; it also can be a collection of impressions or experiences.

It should be noted that some authors (eg Davis, 1984) use the term "concept frame" instead of "concept image". Vinner (1991) uses the concept of 'table' to explain what he means by concept image. He says that when the word 'table' is named several things can be associated to it. For example we can eat at a table, or write and play cards on a table but cannot usually lie on a table. Tables are made by different materials such as wood, plastic, and glass. There are tea tables, computer tables, and poker tables. The images that are brought into one's mind can be contradictory. Students' concept images of a function are different and contradictory depending on the representations they work on. For instance, the expression $y = 5$, represented algebraically, may not be considered a function, because the variable x does not depend on the variable y. Alternatively, a picture representing the same expression graphically may be recognized as a function. In the same way concept images associated with the concept of limits can be developed. Some visual representations associated with the limit concept are the four representations: graphical, numerical, formal and algebraic representation. When the concept limit of a function is named, several images come to mind, such as algebraic techniques, the indeterminate, the graphical representation of asymptotes, the continuity of a function or the sequences of x values approaching a point. Sometimes, but rarely, the definition of the concept is associated with the concept. This illustrates that the concept definition and concept image can be formed independently. Sometimes one of them or both are absent from a student's mind. For instance if the student rote-learned a definition without understanding its meaning, s/he cannot associate any images with the concept. Alternatively, the concept image can be in conflict with the concept definition. If the students conceive of the definition of limits as a point that the function gets closer to as the x value increases, it will contradict the limit of a constant or continuous function. The colloquial meaning of 'close' is near but not coincident (Schwarzenberger and Tall, 1978). Vinner (1991) states that the acquisition of a concept may be made by means of its definition, which is involved in the formation of a *concept image*. However, sometimes the definition is given after the notion of the concept is acquired or is not given at all. Additionally, the teacher's concept image plays a great role in the students' concept formation as Bills and Gray (1999) reported in their research. Their results explain the students' misconceptions caused by the typical exercises and representations that are given by the teacher in the classroom. For example, some students regarded the limit as something that is never reached. This may be a result of examples of functions where the limits are reached not being discussed in the classroom. In addition, the time allocated to algebraic calculations and the amount of limits evaluated in the classroom, led the students to think that the limit concept is encapsulated in the algebraic calculations. Later, when asked to solve exercises they recall the way they solved similar questions in the classroom without thinking of the definition. Vinner (1991) describes four possible situations for the relationship between concept image and concept definition when a student meets a task. He considers each of them as being a cell in the student's cognitive structure, which might be accessed before or after the other, or not be consulted at all when a task is posed. He points out the process where the concept definition is not consulted at all by the student to answer a question as the one that frequently arises in practice.

Concept images and concept definitions have been used as a framework to research students' understanding of various concepts. For instance, Vermeulen (2000) investigated pre-service first-year mathematics education students' concept images after they attended the module Teaching Early Algebra. She concluded that the change of perception of an algebraic expression as a manipulative procedure to a more conceptual understanding did not happen after the instructional component. In the interviews the students exhibited conflicting concept images generated by the spontaneous conceptions and the conceptions developed through instruction. This framework was similarly used in studies to ascertain students' knowledge of the function concept (Vinner and Dreyfus 1989) and the students' perceptions of the continuum (Chesa and Giménez, 1994). Alternatively, other researches used this framework to investigate student's conceptual changes rather than their existing knowledge of the concept. Baldino, Ciani and Leal (1998) and Matsuo (2000), among others, also used the students' concept definitions and concept image to produce a change in the state of knowledge and to investigate the factors that help these changes in the students' limit and real number conceptions and the understanding of geometric figures respectively. This may help improve suitable teaching material for a particular state. On the other hand, Furina (1994) investigated the effect of regular use of spreadsheets on students' concept image of limits. Garcia and Azcáret (1996) also looked for the students' evolution of the concept images associated to a linear function using software.

Students hold spontaneous conceptions, which are acquired through experience in the daily life, when they first face a concept in the classroom. The student's ordinary language, which meanings are most of the cases different from the teacher's one, are not discussed mathematically in the classroom. This does not help the students diminish the intrusion of the spontaneous conceptions in the mathematical language. After teaching, the spontaneous conceptions are modified and adapted to the new knowledge acquired forming the student' concept images and concept definitions. Concept definition and concept images hold by students reflect their understanding of a concept. Therefore concept image and concept definition have been used as a framework to access student's actual knowledge of various concepts or to explore student's conceptual changes.

2.3. THE REPRESENTATION SYSTEM

Douady's (1986) definition of a representation is that it consists of mathematical objects, relations between them, formulations sometimes contradictory and mental images associated with these objects and images. She shows in her article that using different representations for a mathematical concept helps the students associate different mental images to the object of study. Changing from one to another representation the problem is reformulated; new questions related to a particular representation are posed that require using different tools and procedures. Moreover, each representation highlights characteristics of the concept that are not easy to see in other representations of the students' conceptions. For example, nowadays, the computer is a powerful tool for emphasizing the characteristics of various representations at the same time (Garcia and Azcáret, 1996). Schwarz and Bruckheimer (1988) pointed to a higher level of students' understanding when the function concept is first taught through a graphical representation and then transferred to an algebraic representation using a Triple Representational Model. Likewise, Guin and Guzman (1990) used the program LOGO to highlight the various aspects of a function and the relation between them. Confrey and Smith (1992) used the exponential function to illustrate the variation of the students' perception in each representation. Actually, the students perceived, through sketching exponential functions, that they have the same shape and that the points are always bunched together near the origin and more spaced out around the curve. Alternatively, the numerical representation reveals a constant ratio between the values of the function, whilst the calculator is a powerful tool in teaching the algebraic representation. As we can see this interplay between representations concurs with the formation and extension of the student's concept image.

However, to answer similar mathematics questions given in two representations, students might use a certain method within one representation and not use it within another. For instance, Higüeras et. al. (1994) reported the students' use of function proprieties such as increasing and decreasing functions, and continuity to justify the existence of a function given graphically. Nevertheless, the students did not use the same procedure to justify the existence of a function given its algebraic expression. The researchers concluded that the proprieties "Are considered to be graphic characteristics and not analytical ones" (Higueras et. al., 1994, 157). In addition, the use of different representations in the same problem may produce different student answers. Tsamir and Tirosh (1999) found an inconsistency in students' responses using two representations, in the comparison of infinite sets namely the numerical-horizontal representation and the geometric one. Likewise, Furina (1994) reported a discrepancy between the students' responses in the numerical representation and the algebraic one in their evaluation of student understanding of a limit.

While algebraic representation is common to students, graphical and the numerical representations are unfamiliar to Mozambican students. Confrey and Smith (1992) corroborating this view, claimed that the algebraic approach is predominant in textbooks and assessment measures whilst graphs are secondary and the tables even less frequent. In addition, Dreyfus and Eisenberg (1990) pointed to the students' preference to use algebraic procedures instead of diagrams and gave a didactic and a cognitive reason for this preference. This was evidenced by a student, GB, from Williams's (1991) study. She viewed formulas and graphs as useful tools that avoid the need for a conceptual understanding. In a given table of values students answered that if they had a formula or a graph it would not be difficult to solve the task. GB claimed, "*To me, looking at a graph, I mean, you really don't need any concept, you know, sandwich or walking towards a wall or anything, I mean, you just look at it and you can see it*" (William, 1991, 234). I add a fourth position for problems related to limits, namely problems that model a context. Students rarely encounter problems that model reality. Being aware of the advantages of switching between representations for the development of student's conception, Robinet (1983) proposed a didactical sequence. This sequence of lessons helps the students intuitively form the notion of a limit of a function. The students moved from the graphical to the algebraic setting (or vice-versa) analysing the notion from different perspectives, which contributes to the development of their concept image of the limits of functions.

2.4 A DIDACTICAL SEQUENCE FOR THE TEACHING CONCEPT LIMIT

My research questions did not aim to find a didactical sequence for the teaching of the limit concept. However, I found Robinet's (1983) work about the teaching of this concept interesting, because it showed the effectiveness of the change between representations for the students' development of concept images and therefore the conceptualisation of a limit of a function. Therefore, this proposal may be a point of reference for teachers' reflections of more successful approaches to teaching the concept. Robinet (1983) analyses this concept from another perspective. She proposes a didactic sequence for teaching the limit of a function. The students are introduced gradually to the concept through solving different tasks. Using a parabola and a hyperbola the students are guided to conclude algebraically or graphically what is meant by $\lim_{x \to x_0} f(x) = -\infty$ **and** $\lim_{x \to x_0} f(x) = +\infty$. For instance, with the hyperbola they analyse for which value of x (positive) $0 \prec |\mathbf{f(x)}| \prec 10^{-6}$, $0 \prec |\mathbf{f(x)}| \prec 10^{-10}$, $0 \prec |\mathbf{f(x)}| \prec 10^{-n}$, n a natural number as big as we want. The students conclude that if for any n as big as we desire there exists a $x_0 \succ 0$ so that for any x greater than x_0 $0 \prec |\mathbf{f(x)}| \prec 10^{-n}$, we can say that 0 is the limit of the function when x approaches $+\infty$. Or, using the graph, they can conclude that if for any boundary formed by the x axis and a parallel to it, with the width l_0 as small as we want there is a point of the curve M_0 with the abcissa $x_0 \succ 0$ belonging to this boundary so that all points on the curve with abcissa $x \succ x_0$ belong to this boundary, the limit of the function is zero when x tends to $+\infty$. After that the students generalise the formalisation for $\lim_{x \to x_0} f(x) = f(x_0)$ or $\lim_{x \to x_0} f(x) \neq f(x_0)$. In this way, students first deal with concept images of the limit concept using different representations and then formalised the concept.

2.5 MODELLING

Model reality in teaching a concept improves the learning. Often, mathematical expressions for a non-mathematical process or object are created in order to analyse the modelled object or process (Dreyfus, 1991). Boaler (1993) claimed, "The use of examples which students may analyse and interpret, it is suggested, allows students to become involved with the mathematics and to brake down their perceptions of a remote body of knowledge" (p. 118). The benefits of a link between the concept and daily life goes beyond Boaler' s claim in that the link of a concept with an application or as a tool to solve tasks helps the evolution of the students' concept. However, teachers are not used to introducing these types of tasks. Therefore the students feel uncomfortable when faced with tasks related to reality. Philippou and Christou (1994) found out evidence of prospective teachers' failure when trying to produce a word problem for a given mathematical expression including fractions. They cited Simon saying, "They also seemed to be unable to connect the symbolic computations with real world contexts". They do not know how to apply the mathematical concept, mismatching concept images drawn from the object or process model. On the other hand, the solutions to the problems are in some cases incoherent with the context, because students do not contrast and discuss the results obtained keeping the context in mind.

The students' answers to a word problem reveal their conceptual understanding of a concept rather than a procedural understanding. Because I was interested in students' conceptual understanding, in the task of my study I developed a question modelling reality with the aim of accessing students' conceptual understanding of the limit concept.

2.6 CONCLUDING REMARKS

As a lecturer at the Eduardo Mondlane University I frequently observe that the students' concept images of the limit of a function are very weak. The concept definition is absent or very diffuse. The language in the formal definition is very difficult because of the use of quantifiers, modulus and the $\epsilon - \partial$ symbols. I suppose that these factors account for the Mozambican teachers not teaching the formal definition. On the other hand, the definition of a limit as a dynamic process misleads the students to think that the distance between the values of the function and the limit is infinitely small but never zero. This guided the students to a conception of a limit that is never reached. Furthermore, the prototypical tasks used in the classroom contribute to erroneous conceptions (Bakar and Tall, 1992). Translating from one representation to another helps the students to grasp proprieties and characteristics that were not noticeable in other representations. However the students face difficulties when switching from one representation to another and there is a lack of coordination between the different representations. Studies show students' inconsistencies working in different representations (Tsamir and Tirosh, 1999). The algebraic representation was revealed as the predominant in textbooks and assessments (Confrey and Smith, 1992). Evidence has shown that the students are much more confident dealing with the algebraic calculation (Huillet and Mutemba, 1999; Espinoza and Azcárete, 1995; Bezuidenhout, 1999). The difficulties with graphs are related to reading the co-ordinates of a point and seeing graphically that a movement of x toward a particular point on the x axis implies a movement of the function towards a correspondent point on the y axis. Numerical solutions and algebraic calculations represent the same procedure to the students. For them the numerical solution is to substitute the x value and do the calculation as if they are asked to evaluate a limit given a formula.

In conclusion, the students' understanding of the concept of limits of functions is expressed by the definition or/and by images that the students associate with the term 'limits of functions'. Furthermore these images may be represented in different forms. For instance using a graph, numerical sequences or calculations. Doudy's interplay between representations and Tall and Vinner's concept image and concept definition deal with the features mentioned above and explain, to some extent, students' understanding of a concept. Consequently, I drew on Doudy's interplay between representations and Tall and Vinner's *concept image* and *concept definition* as a framework for analysis.

CHAPTER 3

METHODOLOGY

3.1 INTRODUCTION

In this section I will concern myself with the source of data, the procedure used in the collection of the data, analysis of data and methods used in the interpretation of the results.

My intention with this study was to gather information to understand how Mozambican students deal with the concept of the limit of a function, particularly the *concept definition* and their ways of thinking when faced with *concept images* of the limit of a function accepted by mathematicians to answer my research questions. The task and interviews displayed evidences of the students' concept images and definition, and their skills tackling with limit issues. C*oncept images* can only be recalled within a representation. For instance, the concept of a function may be associated with a curve (graphical representation), a table of values (numerical representation), a definition (verbal representation) or an expression (algebraic representation). Therefore, important thing in the study was, which representations the students were able to use, how they related these representations and how they applied the concept in an unknown situation? Their proficiency in working with the representations allowed me to accessing their concept images and their thinking about the limit concept. Two types of data sources were gathered: a task and interviews. The data was triangulated using a teacher questionnaire and interviews conducted in 1998 (Huillet and Mutemba, 1999) and with current participating teachers as portrayed in the following diagram:

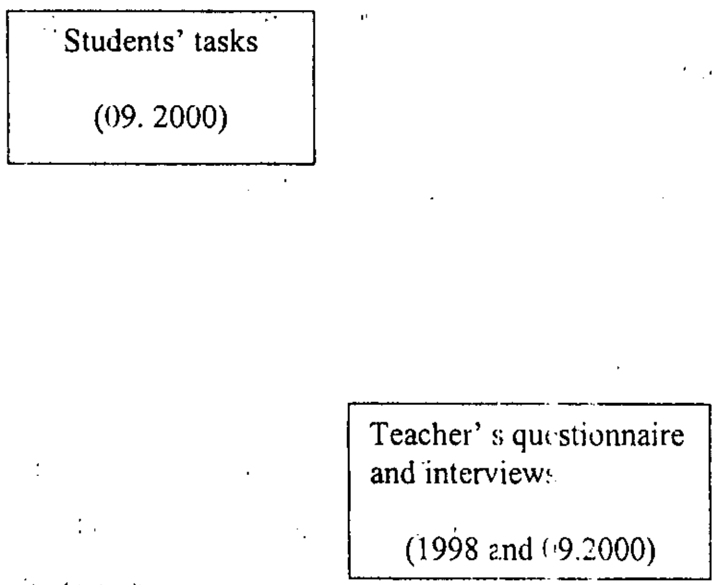


Figure 3.1: Data triangulation of different sources

According to Mathison (1998) the researcher uses triangulation as a strategy to collect evidence, which helps him/her explain the social phenomenon in the study. She claimed that triangulation could bring out not only a convergence between the different data sources but also inconsistency and even contradiction between them. The evidence provided by triangulation - consistent, inconsistent or contradictory - provides the researcher with insight as a basis to construct a theory for the phenomenon. I used the teachers' questionnaire conducted in both schools were I did my research because my experience as a teacher at a secondary school shows that students study mostly from notes taken in the classroom and exercises given by teachers or from examination papers from past years. Therefore it was reasonable to suggest that the formation of their *concept image* is to some extent influenced by their teachers' *concept image*. Besides the teachers' questionnaire I evaluated the data I gathered in the study with other different data sources such as the Grade 12 curriculum, teachers' worksheets and examinations paper. Bills and Gray (1999) reports a relationship between the teacher's representations and the images held by the pupils.

The study was divided into two phases: the piloting phase and the study itself, which was divided into two phases; the administration of a task and the interviews. The pilot phase was the phase where the task was tested in a school in Maputo. The second and third phases took place in Maputo and Quelimane. Students of two schools, one in each city, completed a task and were interviewed.

3.2 THE PILOTING PHASE

The piloting phase took place in one class of 17 students at one school in Maputo. The concept of limits of functions had already been taught in that school. The students were asked to solve the tasks, comment on the language, and style, and format the task.

The developed task showed that the language used was in general clear, but in Question 2 the students were not clear what was meant by the expression practical exercises and in Question 5.2 it was not explicit in the problem that the given formula was applicable as from the moment that the bar was removed from the fire. In addition, the structure applied in some question confused the students. For instance, in Question 3.3 the students were asked to write four sentences using the expressions tends to, converges to, approaches and limit, each at a time, in a non-mathematical context. Some students tried to apply the four expressions in a single sentence other students used only one or twwo expressoions. Thus four sub-questions including one one of the expressions each were introduced instead of Question 3.3. Question 3.1 was improved to providing the students the possibility of explaining what they understood about limit of a function in case they did not remmeber a definition of limit of a function. Accordingly, the structure was modified in accordance with the students' suggestions and difficulties presented. As illustration, the students in the pilot phase had difficulties to sketching the graph of the function $y = \frac{1}{x-2}$. Therefore I changed the structure of the questions. I subdivided the question in two subquestions, specifically I ask them to drawing the asymptotes, to finding the intersections with the axis and then on the basis of the preceding responses sketching the graph. In addition the pilot test allowed me to adjust the suitable time duration to solve the task.

A briefly analysis of the students' responses to the pilot task led me to think that the students' concept definition of a limit of a function was an algebraic calculation, a notation or an approximation and that they conceived of the limit of a function only for discontinuous functions where the limit is never reached. For example students said, "limit plays a role of an asymptote in the graph"; "a limit approaches to a interesting point, but never touch it"; "limit is a point, which the values of the function approach but not equal it". Students seemed to have difficulties in changing from one to another representation as well. Thus, this information provided me with some ideas about the possible students' concept definitions and concept images in the second phase..

3.3 THE STUDY

3.3.1 Mozambicans schools and students

In the second phase a task was administered in two Mozambican secondary schools, one in Maputo and one in Quelimane. These two schools were chosen for convenience. I had already done research about the teachers' conception of the concept of a limit of a function in both schools and I knew the director and the teachers what would facilitate my work.

Maputo is the capital of Mozambique, hence the National Education institutions are situated there and it is assumed that schools resources such as classrooms, books, and other materials are easier to find there. On the other hand Quelimane is a northern city, far from the National Education institutions, where school resources are more difficult to access. Both schools are situated in the city, and teach students from Grade 8 to Grade 12. The students encompassed a range of ages from 17 to 21 at Grade 12. "School A" (Maputo) had approximately 15 Grade 12 classes with about forty students each and six teachers for these students. A class of thirty-six students composed of twenty-two girls and fourteen boys took part in the study. The students were registered in Group B, a science course with the subject Biology. Many of these students had difficulties with the Mathematics. However, there were also some brilliant students in the class. For instance, one of the interviewees demonstrated sophisticated concept images of a limit of a function. He was confident working in any of the representations as well as shifting from one to another representation. In addition, he coped perfectly with the realistic task. "School B" in Quelimane had four teachers teaching 10 classes with about 45 students each. Thirty-five students, thirty-two male and three female, of a class participated in the study. The students were enrolled in Group C, which is a science course with the subject Drawing. The students registered in this group generally performed well in Mathematics, Physics and Drawing. Nevertheless, some students exhibited limitations in working with the limiting process. All the teachers in both schools had the "licenciatura" (Grade 12 plus 5 years at a University).

In Mozambique the concept of limit of a function is first taught in Grade 12 and the students have five mathematics lessons weekly. There has not yet been any Mozambican mathematical study manuals or textbooks issued for this Grade since the Eduardo Mondlane University issued one in 1977. According to Huillet and Mutemba (1999) this manual includes the formal definition of the concept of a limit of a function, theorems related to it and its proof, and a set of exercises that were divorced from theory. In the interviews one out of nine teachers stated that they occasionally used this manual for exercises on calculating limits. The school libraries were inadequately resourced. However, some Portuguese books were available in the schools or in the bookshop. In addition, there are other libraries in these cities, but the students were not used to using these resources. Thus, the majority of the students relied on their notes from the lessons and on teachers' worksheets or past examinations papers.

One class in each school was selected by convenience to respond to the task. Since the teachers influence the formation of students' concepts, my choice to administer task in two classrooms from two different teachers of different cities allowed me to gather a broad range of information about students' *concept image* and *concept definition* of the limit concept. In the last phase I interviewed nine students, five in Quelimane and four in Maputo that had completed the task.

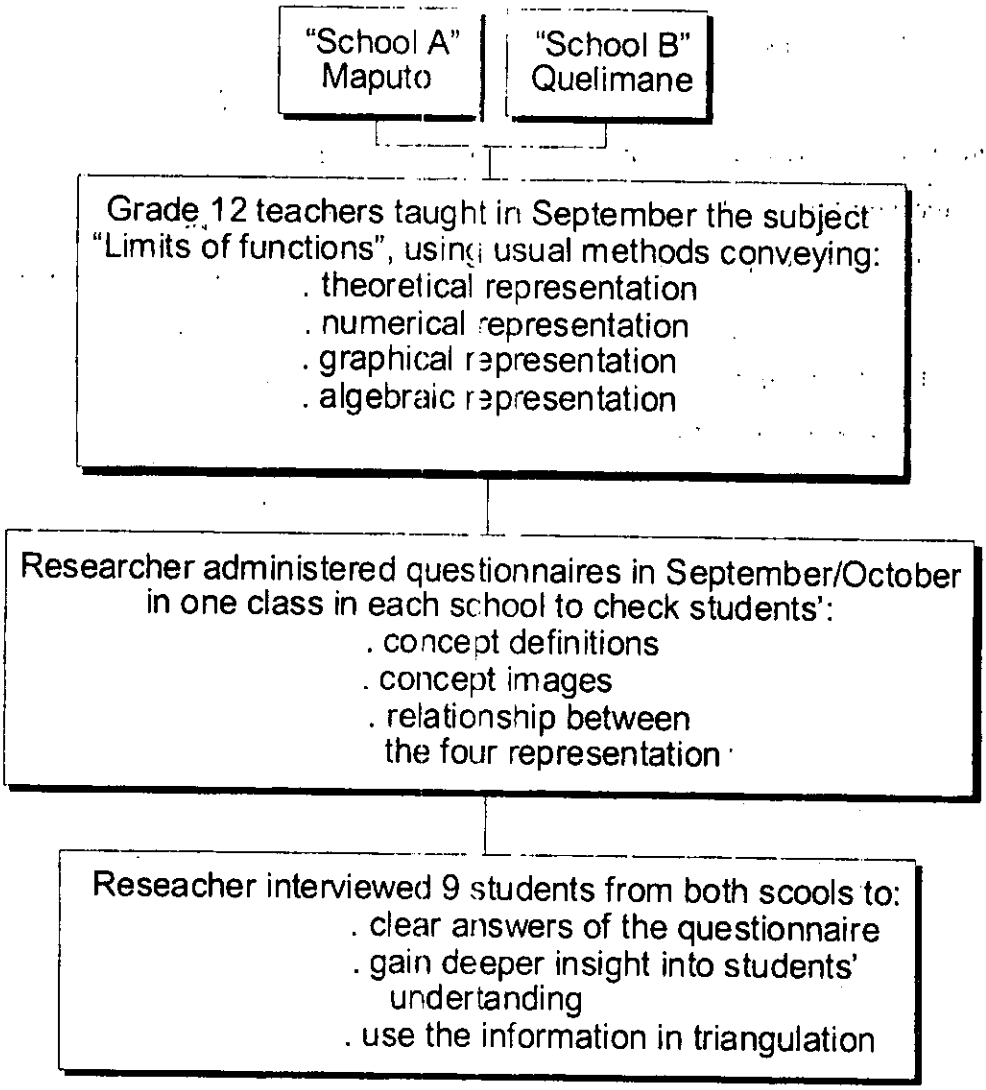


Figure 3. 2: Different stages of the research

3.3.2 The task

The task in Annex A comprised questions on content taught to the Grade 12 students at the Mozambican schools. In addition Question 5.2 had been related to their out of school experiences. More exactly, a question that expressed the behaviour of the temperature change of a metallic bar removed from the fire, which is familiar to them. The questions in the task aimed to help me answer my research questions. A first trial of the task was done in Maputo in April 2000 with some 1st year students from the University and the piloting phase in August in a school in Maputo. Furthermore I discussed the questions with my supervisor. On the basis of this trial, the pilot phase and the discussions with my supervisor I modified the structure of some questions, eliminating or adding sub-questions. For example, Question 3.3 was initial formulated as "What does $\lim_{x \to 4} f(x) = 1$ mean?" This formulation inhibited the students for offering their own explanation of the given notation. The majority of the students merely wrote, "the limit of the function when x gets to 4 is equal to 1". They did not explain what they understood about the notation. Thus I reformulated the question to the sentence, "explain what $\lim_{x \to 4} f(x) = 1$ means to you". Question 6.1 required the students to construct a graph of the function $y = \dfrac{1}{x-2}$ and on review was divided into three sub-questions. The first, asking the students to sketch the asymptotes, the second to determine the intersection with the axis and the third to sketch the graph with help of the previous responses. The task was not anonymous because of the selection of the students for the interviews, but I made it clear that the information would be confidential. The research was conducted in September and October. Initially I intended to administer the task outside of the mathematics lessons in both schools. However in Maputo the teacher gave me the chance to do it during a two-hour lesson. In Quelimane, fortunately, by the time I was there, the teacher was absent participating in a workshop and the task took place during the mathematics lessons. I talked to the teachers and the students involved about the study I was undertaking and what I expected from them. I explained to them that my aim was to understand how the students cope with the concept of a limit of a function using the different representations: theoretical (definition), algebraic, numerical and graphical. In order to achieve this goal, a task and interviews would be undertaken to access students' concept images and definitions of the concept. I made it clear that the interviewees would not necessarily be the best students, but those who gave responses that might be deepened or explained so that I might build an accurate picture of the students' definitions and images. The written test date was set up with the involved students' approval.

The task questions addressed students' definitions and the shifts between representations. They were developed so that I could ascertain whether the students held a static concept of a limit of a function or regarded it as a dynamic process; how they cope with each representation and moved from one to another, and the misconceptions that hinder them to understanding the concept.

As Table 3.1 shows, the questions of the task were grouped in four sets. The first set, that generated the students' concept definitions, was composed by three questions: the definition, sentences applying the terms tends to, approaches, converges and limit, and explanation of $\lim_{x \to 4} f(x) = 1$ (Question 3). On the other hand, the students' concept images were drawn from questions of the following two sets of questions that related the different representations. The second set included the questions that related the numerical representation to the graphical and algebraic representations, namely, Questions 4.2, 6.4 and 7.2. The third set consisted of questions that asked from students' algebraic calculations or related their algebraic solutions to the graph (Questions 6.1, 6.2, 6.6, 6.7 and 7.3). Finally, the fourth set involved a question about a problem of the daily life.

Table 3.1 Distributions of questions that evidenced the students' concept definitions and concept images

Concept definition	Concept image			
	Graphical representation	Numerical representation	Algebraic representation	From context
Q. 3	Q. 6.3	Q. 4.2	Q. 6.1, Q. 6.2	Q. 5
-	-	Q. 6.4	Q. 6.6, Q. 6.7	-
-	-	Q. 7.2	Q. 7.1, Q. 7.3	-

Tables of each set of questions were built in order to seek regularities and discrepancies. These tables facilitated the materialization of the students' concept definitions and concept images. Some of the categories I used were taken from the literature. I looked for students' expressions or strategies from the data that led me to generate new categories so that I could build a theory. Following is the rationale of the items included in the task.

Question 1

With the general questions I wanted to find out what beliefs about Mathematics the students held and how they prepared themselves for the mathematical lessons.

Question 2

From task and interviews that I have administered to Mozambican Grade 12 teachers in 1998 it was assumed that the lessons on the topic of limits of functions were used essentially to calculate limits algebraically. Both the definition and the tasks related to everyday life were not usually dealt with in the classroom. With this question I wanted to extract the students' feelings about the emphasis of each representation in the classroom.

Question 3

It seems that students have difficulties with the definition. The answer to this question helped me to understand the role of language (3.1, 3.3, 3.5) and misconceptions developed from everyday meanings for some words (eg. tends to, approaches) (3.4). Furthermore I could identify which definition the students were more likely to give – the formal definition with $\epsilon - \partial$ symbols or a dynamic answer. I extracted this question from the Monaghan (1991). However the idea belongs to Cornu (1980).

Question 4

The students normally use the mechanical substitution of the x value as if they was working in a numerical representation and have problems transposing points to the graph. The purpose of question 4 was to challenging the students (4.2(a)), seeing to what extent they related the graphical representation with the numerical representation and the definition of the limit of a function (4.2(b), 4.3). In addition it might enable me to see if the students recognized the limit of a function as a dynamic process.

Question 5

The answer to question 5.2(c) showed the students' proficiency applying the concept of limit of a function in an unfamiliar situation. Furthermore the questions 5.2(a) (b) (c) (d) enabled me to compare their logical reasoning and the mathematical procedures they used. The correct response to question 5.2(e) could probably show the students' understanding that $\lim_{x \to 6} T(t) = T(6)$

Question 6

The answers to this question gave insight to how the students dealt with the construction of a graph and if there was coherence between the answers on the basis of the graph and the results of the calculation (6.1(a) and (b)). In addition I wanted to verify if the students had an accurate notion of asymptote and how they tackled the notion graphically, algebraically and numerically. In response to question 6.6(a) the students had to verify whether the number, which annuls the denominator, was really an asymptote. In addition the students demonstrated if they recognized the asymptotes of the function, $f(x)$, and how they related these limits (definition of limits) to the graph. With question 6.7 I wanted to ascertain to what extent the students recognized the asymptotes and dealt with the information of an unknown function.

Question 7

The emphasis in most lessons is placed on algebraic calculation. Therefore it was thought that students had few problems calculating the limit of a function. With this set of calculations I wanted to observe how students in general tackled the techniques and how they dealt with the zero as numerator and as denominator of a function (7.1(b) and (c)). Mamona-Downs (1990) claims that the zero is not a limit problem. In question (7.2) the students' ability to calculate the limit of a function and to connect the algebraic representation with the graph was most important. The varieties of the limits calculations were taken out from the different teachers' work sheets.

3.3.3 The interviews

The interviews were conducted during September and October 2000 with duration of a half an hour for each student, in the afternoon in Quelimane and in the morning in Maputo. Students in Maputo left their lessons while those in Quelimane offered time after school. In both schools they took place in a classroom. I made an appointment with each student for the interview in advanced for an hour that was convenient for the student and for me. I told the students how long it would take and guaranteed that the data will be used only for the study and their names would not appear in any case.

For the interviews I prepared general and particular questions related to each student's answers on the task where the students showed the strategies used to solve the tasks, so that I gained access to the students' thinking and a fuller picture to clarify certain aspects of the responses in the task. In the beginning of each interview I tried to transmit confidence to the students asking them general questions such as if they like mathematics, which course they wanted to follow. In addition they were told to be free to say if they did not understand a question. I picked out the following extract of an interview as an example.

I: "good morning"

P: "good morning"

I: "we did for two days a written test. How did you feel, were you confident about what you wrote?"

P: "yes, I think I did well"

I: "do you like mathematics?"

P: "yes"

I: "which course do you want to follow?"

P: "civil engineering"

I: "why did you choose this course?"

P: "because I like physics and mathematics"

I: "so, good luck, we will see us next year at the university. As I explained in the first time with the interview I would like to understand some aspects that was not clear in your responses. Before, I want to say that you are free to telling if you did not understand a question, so that I repeat it"

Maxwell (1996) raises an issue related to interviews, namely the inaccuracy or incompleteness of the data. He added that to avoid these pitfalls and enhance the validity of the study the researcher might record or videotape the interviews and transcribe them. All but one interview in the study was tape-recorded. The reason for not recording one of the interviews was that the researcher realised at the end of the interview that the recording button had not been pushed. This was the first interview with Salimo, and my supervisor observed it. During the analysis of the interview, he suggested immediately write down what I remembered about the interview. In addition he gave me advice about some aspects to improve my interviewing. My skill was improved whilst I interviewed the students. Throughout the interview I paid attention to the students' procedures and their understanding of the terms used in the limiting process. The interviews were based on the questions that have not been answered clearly enough in the students' tasks. These included the definition of the concept of a limit, the meaning of the terms "tends to", "limit", "converge" and "approach" as well as the questions relating the three representations. In addition I tried to understand which strategies the students used to move from one representation to other based on the task questions and the justification of their reasoning or how they had performed some tasks. I kept notes made by the students for further analysis. Further questions that required the interviewee to discuss issues they mentioned before were addressed to each student, and I took notes about the way in which the students coped with the tasks during and after the interviews.

The tapes recorded in Maputo were transcribed immediately after the interviews. Unfortunately, the same was not possible in Quelimane where I did not have access to a computer. The main aspects of the students' definitions and images emerged from the interviews were briefly annotated after the transcription of the interviews.

3.4 ANALYSIS OF THE DATA

3.4.1 The tool used

In the analysis of my research I used a word processing program. The tape recordings of the interviews were all stored in a file. Summaries and tables of the students written answers were processed as well as the report of the study. This facilitated my work because it was possible to moving words, sentences or paragraphs from a place to another within the text, correcting spelling errors, sorting and retrieving data. However, I did some activities without using the computer. For example, I coded the written task and developed the categories.

3.4.2 Analysis of the written responses

In this section I will describe the procedures of the analysis of each set of written questions. I assigned a code, written on the left margin, to similar responses from students. I based the generation of these categories on Miles' (1984) work. Some of the categories such as dynamic, boundary, and static were derived from the literature, while other categories such as notation algebraic calculation, asymptotic, barrier and formal definition emerged from the data. The analysis of the written responses was made according to the set of responses I mentioned before.

The students' definition

The definition of the concept of a limit of a function is influenced by the definition of the terms tends to, converges to, limit and approaches to used herein. On the other hand an explanation of the meaning of the notation $\lim_{x \to a} f(x) = b$ may reveal what definition the students hold. Therefore, I analysed the students' definitions, sentences applying to the terms tends to, approaches, converges and limit, and explanation of $\lim_{x \to 4} f(x) = 1$ together. Firstly, I will explain how I analyzed the students' responses to the definition. My initial list of the students' definition included eight categories: notation, algebraic calculation, extreme, continuity, relation, formal definition and decreasing/increasing function. These categories were reviewed after thorough scrutiny and a discussion with my supervisor. In the second list the category 'relation' was replaced by the category 'static because the isolated representation of that category was better described by its static nature than a relation between the variables *x* and *f(x)*. In addition a new category, the dynamic category was introduced to include definitions using the terms tends to and approaches. The definition "I understood that the limit of a function corresponds to the value that the image of a number *x* assumes when *x* approaches a finite value, on its left or on its right" was an example of this category. In the final list, the definitions were divided into two main categories: the static and dynamic categories. The static category was sub-divided into notation/algebraic calculation, boundary and formal definition. On the other hand, the dynamic category includes the continuity and approach. In this list the old categories notation and algebraic calculations fused together because they seem to represent the same concept definition. As the analysis proceeded, it became apparent that the evidence for some categories was insufficient strong and there existed some contradictory evidence. In this case I rethought of the answers and placed them in the existing categories or introduced new categories where these characteristics best matched. On the other hand, due to difficulty with the English language, there were some categories that made sense in Portuguese but did not have the same meaning in English. Accordingly, some categories such as 'extreme' disappeared and other fused together to produce new categories. 'Boundary' and 'approach' were new categories belonging to the last list.

The data from this task was triangulated with data from the teacher questionnaire and interviews conducted in 1988 and reported by Huillet and Mutemba (999). According to Mathison (1988), triangulation controls bias of a unique source of data and establishes valid evidence so that explanations of the phenomenon can be constructed. She added that the evidence might be convergent, inconsistent or contradictory. For example the evidence sources concerning the students' definition converged to the category notation/ algebraic calculations through triangulation. Teachers dedicated lots of time handling algebraic calculations and questions about limits of functions in the examination papers only required algebraic calculations (Huillet and Mutemba, 1999). Therefore the students' concept definition as a notation or an algebraic calculation reflected their experience in the classroom.

Secondly, the students' understanding of terms related to the limit concept (eg tends to) prejudiced the students' formulation of the definition. The first analysis of the students' sentences provided the following categories: approximation, direction, constraint, frontier, resemble, target, inclination, change, location and capacity. These categories were refined throughout the analysis and as result some of them disappeared and others fused together generating new categories. The final list included the categories approach, constraint, boundary, tendency, direction, location and aim.

Thirdly, the explanations of $\lim_{x\to4} f(x) = 1$ were assigned to the categories algebraic calculation, equation, dynamic and boundary. In the first list I introduced the category reading, but it was replaced by the category dynamic because in those explanations the students used the verbs tends to and approaches that highlighted a motion in the concept.

The shift from the numerical representation to the graphical and algebraic representations

Questions 4.2, 6.4 and 7.2 related the numerical representation to the graphical and algebraic representations. I looked for students' strategies when performed these tasks. They used two strategies, one was the use of a formula and the other was a looking for the behaviour of the function to guess its limit. In the first case the students attempted to find a formula which help them to calculate the required questions algebraically.

The shift from the algebraic representation to the graphical representations

The set of questions that related the algebraic calculations to the graph (Questions 6.1, 6.6 and 7.3) supplied evidence of the approaches employed by the students to solve the questions. In the analysis I was concerned with the students' strategies for transposing the algebraic limit results to the graph.

Initially, the work on the representation data produced the categories dynamic, static, value of a function and asymptote. In the final categorizing the data were split into dynamic and static categories. The static category was subdivided into three categories, specifically, value of the function, boundary and asymptote.

The limit in a context problem

The task included a realistic problem. For the analysis of this question I was interested in the students' ability to linking the mathematical procedures with their experiences outside the classroom. It means, I wanted to see how consistent were their concept images resulting from the given context and those resulting from the mathematics. In addition, I looked for the strategies they used to answer a question that might be responded without resorting of mathematical procedures.

3.4.3 Analysis of interviews

Nine students were interviewed with the following goal:

- to ascertain what definition of limit of a function they held
- to access the students' thinking about the concept of a limit of a function
- to observe how the students work on a representation and their ability to move from one to another representation
- to discover other misconceptions that hinder the students' understanding of the concept

The interviews were audiotaping and were transcribed immediately after the interviews or as soon as it was possible. To transcribe the tapes I used a word processor. Remarks I made during or after the interviews and that made by the students were used in analysis. Firstly, the interviews were analysed one at a time, and then compared each to another. Similarities and differences were used to categorize the data. The students' responses were coded and themes and patterns were emerged throughout the reading and re-reading of the transcripts and the notes. The codes were written in the left margin of the interview transcripts. The emerged categories and relationships were added to the already existed before the interviews. Tables from the interviews were constructed, so that the categories come into sight.

These tables helped me to identify and interpret the students' ability in working within each representation, in moving from one representation to another and consequently the students' concept images.

3.5 ETHICAL ISSUES

Before and during the study I concerned myself with the ethical aspects of the research. I took in consideration some aspects pointed by Bogdan (1984) such as the identification of the researcher, the research procedures, subjects' privacy and confidentiality and not expose the subjects to harm. Accordingly, I asked the pedagogical director of School A in Maputo and the director of School B in Quelimane for permission to undertake my research. They were explained about the aim and the expected outcome as well as the instruments to be used in the study. Then I was introduced to the two teachers involved in the study. I talked to them about my intention with the research and granted that their privacy and confidentiality as well as of the students would not be violated. Mitchell (1992) raises the responsibility of the Psychologists for the welfare of their subjects and protection of their human rights. Although the subjects of the study were the students, their teacher might be blamed for the poor students' performance. Thus, to keep their welfare it was important to promise them the anonymity of the students in the report, but it was not possible to eliminate the risk that the students form opinions about their teachers because of the issues they were not used to tackle in the classroom. The teachers were present in the first time I met the students and the explanation about the research purpose was repeated to the students. In addition it was explained that the task was not to evaluate their skills and would not be marked and did not influence their mathematics marks. The students participated voluntary in the study. Although the task was not anonymous because of the further step of the study, I promised the students not using their names in the study report. Therefore I used pseudonyms herein. In the interviews I tried to assume a non- interventionist position, in order to avoid bias in the data.

CHAPTER 4

DATA ANALYSIS: OVERVIEW

4.1 INTRODUCTION

In learning mathematics students are frequently faced with new mathematical objects or processes. When they hear or think of a mathematical object or process, they associate it with other previous experiences and cognitive constructs. In other words, they associate them with mental pictures (Tall and Vinner, 1981), representations (Douady, 1986) or facets (DeMarois and Tall 1996). Kaput argues that the process of materializing these mental images relies on a representation system (in Dreyfus, 1991). When the student encounters a new concept s/he forms a concept image of the concept under consideration. Concept images are the individual mental associations someone makes to the name of the concept. Besides mental pictures related to a concept, the concept can be specified using words. This is a concept definition of the concept (Tall and Vinner, 1981).

In this chapter I will present an overview of the students' concept definitions of the concept of a limit of a function (Section 4.2). In Section 4.3 I will concern myself with the students' concept images and how they represent limits within the numerical, graphical and algebraic representations. Along with this, I will discuss the students' ability to solve tasks modeling reality (Section 4.4). The aim of this chapter is to portray the students' responses to each question of the task and the interviews, placing them in categories within the notions of concept image and concept definition. Furthermore, the analysis of the students' work within the different representations will produce concept images associated to each representation.

4.2 THE MOZAMBICAN STUDENTS' CONCEPT DEFINITION OF THE LIMIT OF A FUNCTION

A *concept definition* is a definition, in words, created to specify a concept (Tall and Vinner, 1981). I categorized the students' definitions of the limit of a function, given in the task, into seven categories. These categories emerged from the students' answers to Question 3.1, that asked them to give a definition or an explanation of the concept of a limit of a function (see Appendix A). Table 4.1 displays the categories that emerged from the students' answers to this question.

Table 4.1 Distribution of students' concept definition

CATEGORIES		STUDENTS	
		Quelimane (35)	Maputo (36)
No answer		7	12
STATIC	Notation/ algebraic calculation	12	13
	Boundary	8	1
	Formal definition	3	3
DYNAMIC	Continuity	1	-
	Approach	4	7

An explanation including examples of students' work and a detailed analysis of the nature of the concept definitions of these categories will be undertaken in Sections 4.2.1 to 4.2.6. As portrayed in Table 4.1 different students' concept definitions emerged from the task, some of them incorporating a static character and others a dynamic one. The definitions classified as dynamic suggested an action, an idea of movement of the points of a function towards a value. Most of the time this process was conceived as unfinished. Alternatively, the static nature of the definitions was noticeable by the conception of a limit as the product of a process, as a fixed value that is obtained through successive mathematical operations or as a boundary. This boundary might or might not be attained. From the definitions with a static character, where the limit concept was regarded as a fixed number, three categories, namely the *formal definition*, *boundary*, and *notation/ algebraic calculation* were distinguished.

4.2.1 Notation/ algebraic calculation (NOT/CA): the limit of a function is $\lim_{x \to a} f(x) = A$ or a limit of a function is the substitution of the x value in order to obtain the y value.

Table 4.1 shows that twelve (16,9%) students from Quelimane and thirteen (18,3%) from Maputo tried to use either the notation or the algebraic calculation as a definition of the limit of a function. For example,

"$\lim_{x \to 2} f(x) = 1$ means that certain values from $-\infty$ to 2, being substitute in the x place, might be after a solution equal to 1"

"limit of a function is a method we use when we want to calculate a number, non exact but approximated"

"limit of a function are limits that to evaluate we must apply rules, sinus, co-sinus, tangent and cotangent"

"limit of a function is $\lim_{x \to \infty} \frac{3x-1}{x+1}$. what means what is the value of this limit when x tends to infinity".

"limit of a function is the value the function assume when x is attributed the value $x = a$"

The key words of this category were "substitution", "attribution", "method", "rules", "evaluation", "calculation", "solution" and "number". These words were used within a context of algebraic calculation. The variable x of the function $f(x)$ was substituted or attributed a value to which x approaches, then algebraic procedures using known mathematical methods (L' Hospital, factorization, conjugate) and rules (cancellation, notable cases, trigonometric identity) were undertaken to evaluate the limits and to obtain a number as a solution. The last example demonstrates explicitly that the students associated the limit concept to a manipulation of rules and procedures. It may be that the prevalence of students' concept definition as a notation/ algebraic calculation has its origin in the students' practice in the classroom. The number of limit calculations tackled and the time devoted to them in the classroom was evident in the students ordering of Question 2 items. That is, the students ordered their activity from less classroom time to more time, as theory, application of limit in the daily life, graphs and algebraic calculations. This is illustrated in Table 4.2. Therefore, students spent most of the time in the lessons, under the topic "Limits of functions and Continuity", determining limits.

Table 4.2 Students' scale of the time spent on aspects of the topic "Limits of functions and Continuity"

	1		2		3		4	
	Q	M	Q	M	Q	M	Q	M
Theory	13	18	11	9	7	3	1	4
Calculations	6	4	2	5	5	5	18	20
Problems	2	8	12	17	11	7	6	2
Graphic	12	-	7	8	3	18	9	8

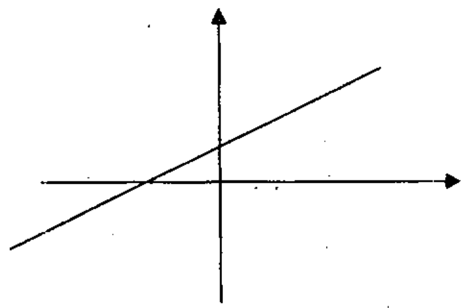
Q – Quelimane

M – Maputo

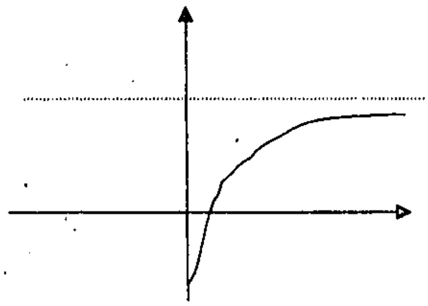
The table shows that the two groups agreed that theory received the least attention and algebraic calculations the most, while everyday problems also received little attention. Notwithstanding, the students' opinions diverged concerning the graphic representation. Whilst the students from Quelimane thought they had spent less time with this issue, the Maputo students' responses seemed to attribute sufficient time to it. In addition, Irene, a Grade 12 teacher, corroborated the students' view, when interviewed in 1998 (Huillet and Mutemba, 1999). She stated that, "*Now, after our talk, I see that it is possible to challenge the students with nice tasks relating the different representations or exercises modeling reality. I have already seen some nice problems in some books. Despite that, I cannot waste time with those problems because I must prepare the students for the national examination. There, the students are only asked to calculate limits*". Nadine, another Grade 12 teacher added, "*We do not have enough time to relate limits of functions with graphs, the students must know the rules and methods of limit calculation*" (Huillet and Mutemba, 1999). Similarly, Tall (1992) and Bezuidenhout (1999), among others, stated that many students prefer algebraic manipulation in handling the limiting process. Accordingly, it was reasonable that the students conceived of the limit as a *notation/algebraic calculation*, an image with which they were faced with frequently.

4.2.2 Boundary (BUN): limit of a function is the maximum or minimum value that the function assumes.

The Portuguese word "extremo" signifies a line that marks a limit, a dividing line. The students used it to mean the greatest or least value of the function. On the other hand, the students held an ordinary sense of the term limit as a barrier, a frontier, and an end. They may think of a line, which is impassable. This line can eventually be reached but is not surpassed, because it is the end, frontier or boundary. So they transferred this non-mathematical language, or spontaneous conception (Vinner, 1991) of the word limit, to the classroom and defined the limit of a function as a y value, which cannot be crossed, as a boundary: maximum or minimum value, which is meant as the greatest or the least value of the function. The expression 'last value' of the function was common in the interviews. For 'last value' the students meant the greatest or least value of the function. Figure 4.1 gives a picture of this conception.



4.1a)



4.1b)

Figure 4.1 Illustration of the term 'last value'

In Figure 4.1a) when the sequence of x values gets closer to 2 the matching sequence of y values gets closer and closer to 5, reaching it. Alternatively, the limit of the function in Figure 4.1b) is not attained. The sequence of y values approaches the limit 3 as a line that is never reached. Therefore 5 and 3 were considered the last value of the functions, the limits. Henceforth, the phrases last value, and maximum or minimum value of a function will be used in the sense of the closest term below or above a boundary. Therefore the boundary might be regarded as a point (Figure 4.1a)) or as a line (Figure 4.1b)). The mental image of a limit as an unsurpassed boundary is corroborated by Cornu (1983), who claimed that most of the time the limit is viewed as an "impassible limit". I borrowed the *boundary* category in the analysis of students' concept definitions from Davis and Vinner (1986) who used it with a similar meaning. In my study, nine (11,8%) students held the concept definition of a limit as a boundary. For example

> "Limit of a function is the maximum value, where the function ends"
> "Limit of a function is the limit maximum, where the function ends"

"It is the maximum value of a function if this function tends to a certain value"

"Limit of a function is the number finite or infinite where the function attains its maximum or minimum point"

What the students tried to say was presumably that the function couldn't assume a value that was greater (lower) or equal to this limit. It means that the limit is the last value, the biggest or the smallest, that a sequence of y values might or might not attain. This category is evidenced by the students' use of the words "maximum and minimum value", "where the function ends" to indicate that the function did not go further than this last value. To add to these two concept definitions, six students conceived of the limit as a formal definition.

4.2.3 **Formal definition (FD)**: definition with $\varepsilon - \delta$ symbols:

It was not my intention to concern myself substantially with the formal definition. However there was a notion of the $\varepsilon - \delta$ definition evident in at least six students' concept definitions. These students attempted to formulate a formal definition, but only one wrote a complete sentence:

> "Limit of a function is understood as a point $x = a$ in the interval of the domain of the function, so that exists a number $\varepsilon \rangle 0$ it is $\varepsilon \rangle a$ where $f(x) \rangle f(a)$, so it means limit of a function when x gets to a is equal to A, i. e. $\lim_{x \to a} f(x) = A$".

The other two students seemed to have heard about this definition, although they only retained the symbol of quantifier, $\forall$ (para qualquer que seja), and the ε symbol, as we can see in their answers.

> "$\lim_{x \to a} f(x) = b, \forall \varepsilon \rangle 0$"

"$\lim_{x \to a} f(x) = b$; para qualquer $\forall$"

Severino, the first student, remembered the number ε and the use of the symbol greater than, but his definition is not understandable. The Mozambican syllabus suggests the introduction of this topic using the formal definition at Grade 12, but it calls the attention to the abstractness and the difficulty of this definition (Ministério de Educação, 1993). The Mozambican teachers rarely give the formal definition because they are aware of the students' difficulties. Severino's teacher affirmed in the interview with him that he did not give this definition to the students because they would have two new aspects to understand, the new variables (ε and δ) and the abstract language using quantifiers and those symbols. Other researchers have also reported student difficulties with quantifiers and the ε - δ symbols. For instance, Tall and Vinner (1981) and Vinner (1991) ascribed the failure to understand the formal definition to the quantifier and symbols used in it. Accordingly, it is understandable that those students faced difficulties to recall this definition. Two other students gave definitions including the words "interval" and "neighbourhood", expressions suggested by Tall (1991) as showing the static nature of the limit of a function.

> "Limit is to calculate a value which is in the neighbourhood of a certain number *a*"
>
> "Limit of a function of a real variable defined in a closed interval $[c, d]$, where the point a belongs to it: $c \langle a \langle d$"

The students' words "interval" and "neighbourhood" insinuated that the distances between the function values after a certain term and the fixed number *a*, are very small, which seems an attempt to use the $\varepsilon - \delta$ definition. These three categories all described concept definitions with a static character. Alternately, twelve (16,9%) students' definitions suggested a dynamic nature. Accordingly, two other categories were developed, the *approach* and the *continuity*. In the *continuity* concept the limit was used as a tool to determine the continuity of the function. The motion idea of the function points was noticeable in this category by the use of the words continuous and tends. Similarly, the concept definition, *approach*, contained the verbs tends to and approaches, both of which give an idea of motion.

4.2.4 Continuity (CON): the limit of a function is A when x gets to a if it is continuous at $x = a$.

One student assigned the limit as a way of analyzing the continuity of a function. The translation of his sentence was:

> "Limit of a function is a means of defining if a function is continuous. If the function is not continuous the limit does not tend to a real number on the right or on the left"

The dynamic nature of this concept definition is substantiated by the motion idea of the words continuous and tends to. The student who stated that the limit was a means to examine the continuity of a function, conceived of the definition of the concept of a limit of a function as a *tool* of solving posed tasks, in this case, the continuity of a function. According to Douady (1986) a mathematical concept is considered an *object* when its definition and proprieties are focalized. For example, the limit is considered a mathematical object in the case when its definition, the theorems linked to it, and the rules and methods of evaluating a limit are learnt. Douady (1986) distinguishes *objects* from *tools*. *Tools* are when we use a concept in the process of solving tasks. For instance, we use the concept of a limit of a function to verify the continuity of a function or to determine the asymptotes of a function. The student had applied the algebraic calculations of limits to examine the continuity of the function and the intervals where it decreases or increases. Accordingly, it seems that what s/he remembered about the topic was where this concept was applied. This leads me to consider an observation I have made during my lessons as a teacher. The students do not consider the theory important. For them what seems to be important is how to apply the formulas in the practice exercises. Besides this procedural aspect, other students had the concept definition of a function as an approach towards a point that is unfinished.

4.2.5 Approach (APR): the sequence of y-values approaches to b, when the correspondent sequence of x-values gets closer to a.

As was said in Section 4.2.3, due to the difficulties the students met when faced with the formal definition, both teachers did not teach it. Instead, they introduced a definition, considering the limit as a dynamic process. As the examples portray eight students tried to recall the learnt definition:

> "Limit of a function are the values that the function more and more approaches to, when the variable x approaches a"
>
> "I understood that the limit of a function corresponds to a value, which is an image of a number x when x approaches a finite value, either on his right or on his left".

Eight students employed the term "approaches" to assign a motion nature to the concept. The additional expressions "more and more" emphasized this idea, and conferred, in addition, a continual movement towards the limit.

Summarizing, the students' definitions were divided into two broad categories including concept definitions that were dynamic or static in nature. The static concept definitions were subdivided into three categories: the *notation/ algebraic calculation*, the *boundary* and the *formal definition*. Their static nature was supported by words that supposed a fixed number or value. or a distance. For example, the students used expressions such as interval, neighborhood, maximum and minimum or last value. Alternatively, twelve students (16,9%) defined the limit in a dynamic sense, a dynamic process where y gets closer and closer to b. The dynamic concept definitions were *approach* and *continuity*. The verbs converges to, approaches and tends to used in these definitions gave the notion of something that moves continually, which may or may not reach the target. The expression 'more and more' in some of the definitions stressed the idea of a process in progress.

Doing a simple count we see that, 33,3% of the interviewees conceived of the limit as a dynamic process (*approach*), 55,6% as a static number (*boundary* or *notation/ algebraic calculation*), and as points through which the function passes (11,1%). Besides the concept definitions a student holds, s/he develops concept images in the learning process and as a result of his (her) own experiences outside the classroom. These concept images are evidenced by the students' switch between the different representations. Accordingly, in the next section, an overview of the students' concept images and how they worked with the different representations will be presented.

4.3 THE STUDENTS' CONCEPT IMAGES AND THE THREE REPRESENTATIONS: ALGEBRAIC NUMERICAL AND GRAPHICAL

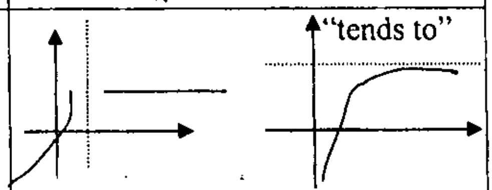
4.3.1 THE STUDENTS' CONCEPT IMAGES

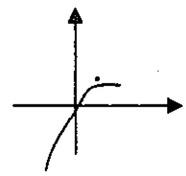
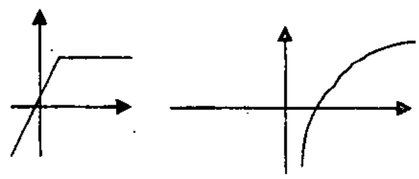
Concept images are the individual mental picture associations with a concept. Dreyfus reporting Kaput's theory affirmed that, "the act of generating a mental representation, relies on a representation system, that is, concrete, external artefacts that can be materially realized" (Dreyfus, 1991, 31). These may be words, graphs, and symbols. For instance, the concept of limit of a function may be associated with the ε and δ symbols (algebraic representation), with a curve (graphical representation) or a sequence of values of a function going closer to a certain value (numerical representation). In addition, Douady's (1986) definition of a representation is that it consists of mathematical objects, relations between them, formulations that might or might not be contradictory and mental images associated to these objects and images.

DeMarois and Tall (1996) used the word facet and described it as "ways of thinking about a mathematical entity and communicating to others, including verbal (spoken), written, kinaesthetic (enactive), colloquial (informal or idiomatic), notational conventions, numeric, symbolic and geometric (visual) aspects" (DeMarois and Tall, 1996; 298). The different approaches of teaching a concept using diverse representations helped the students to build up a concept. Actually, the use of diverse representations allowed the students to analyse the mathematical objects from various perspectives, using different procedures, rules and methods because each representation is associated with specific procedures and rules and highlights specific characteristics of the object. However, the students may obtain different answers to the same mathematical problem working within different representations (Arcavi, Tirosh and Nachmias, 1989). It may be a point of conflict if a student holds different images of the same mathematical object and recalls them simultaneously (Tall and Vinner, 1981). This happened to a student, Gomes, who gave two different responses to a question about asymptotes of a function when working within algebraic and graphical representations respectively. However, he did not notice the contradiction in his responses. The representation system is a device that influences how one' s mental images of an object or process come into being (Dreyfus, 1991). Moreover, the students' ability to change from one representation to another relies on their images of the concept in consideration. That is why I will discuss the two concepts (representations and concept images) together in this section. As with the concept definition, the students' concept images were split into dynamic and static images. These two categories are the same as those used in the analysis of the students' concept definitions. Students conjured up images when engaged with words related to the limit concept. For instance, Cornu (1991) pointed to the expressions "approaches but cannot reach", "cannot pass", "tends to" as concept images associated with a dynamic process of "getting close", "growing close" or "going on forever". Alternatively, the expression "limit" suggests a "boundary point". I classified the concept images *asymptotic* and *motion picture* as dynamic images, and the concept images *value correspondence*, *procedural* and *barrier* as static images (see Table 4.3). It is to note that the graph representing $\lim_{x \to +\infty} f(x) = 3$, a dynamic concept image, interpreted a boundary concept definition, considered static, in Section 4.2.2 as well. The reason for an apparent contradiction is that, whilst in the boundary concept definition the asymptote represents a line that is never surpassed, in the asymptotic concept image the asymptote represents a target towards the function continual approaches.

The task and interviews evidenced these five essential aspects of the concept of a limit of a function conceived by the students.

Table 4.3 Students' concept image categories and the respective image

CATEGORIES		STUDENTS	IMAGES
Dynamic images	Asymptotic	28	 $\lim_{x \to 2} f(x) = 5$ $\lim_{x \to +\infty} f(x) = 3$
	Motion picture	5	Values of y going closer to a point $\lim_{x \to +\infty} f(x) = 2{,}99..$ $\lim_{x \to +\infty} f(x) = 3^-$

CATEGORIES		STUDENTS	IMAGES
Static images	Barrier	6	 $\lim_{x \to 2} f(x) = 5$
	Value correspondence	25	 $\lim_{x \to +\infty} f(x) = 3$ $\lim_{x \to 2} f(x) = 5$
	Procedural	20	Working with algebraic expression or regularity

4.3.1.1 Asymptotic: the limit is considered as a value that is never attained, as an asymptote.

In Question 6.3 the students were required to give the limits of the functions presented in a graphical representation. Among them, the graphs represented in Figure 4.2 were included.

a) b)

Figure 4.2 What are the limits of these functions?

A number of students argued that the limit in Figure 4.2a) was different from the limit in Figure 4.2b). They stated that in 4.2a) the limit was 3, but claimed that 3 was not the limit in Figure 4.2b), though in mathematical sense the two limits were equal. The reason for rejecting the answer was that, *"The limit is not equal to 3 because the function approaches more and more to this value but it did not assume it".* Alternatively, they said that the limit of this function tends to 3. Pedro stated in the interview, *"This is not absolutely the value, it attains 3 by rounding it up"*. Arthur added, *"The function approaches 3, does not attain it, so the limit is* 3^-*"*. Several students used the symbol 3^- meaning that the limit is not reached.

It appears that the students' concept image of a limit of a function is frequently linked to the asymptote. Their conception of an asymptote was a straight line (a point or points) that was (were) not touched. It indicated that the graph approached this line (point or points) but never touched it (them). Similarly, the ordinary word *tends to* associated to the limit concept suggested an image of continual approach towards a value, that is never reached. For example, Pedro stated, that,

> "Tends to means that the values of the function approach to a certain value but does not attain it".

This assumption explains why some students tried to sketch the graph given the condition $\lim_{x\to 2} f(x) = 5$, with asymptotes (Figure 4.3).

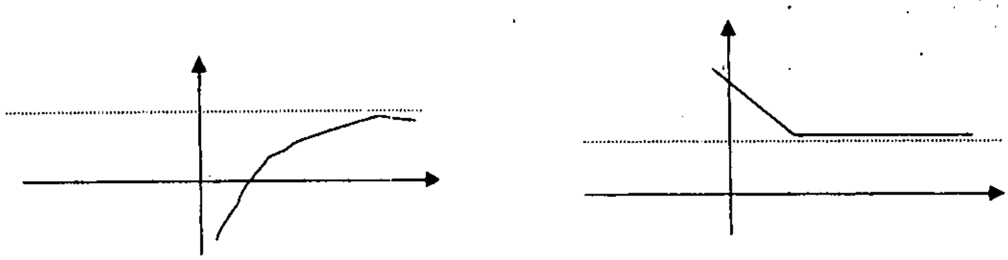


Figure 4.3 Students' sketches of $\lim_{x\to 2} f(x) = 5$

As we see in the two examples above, $y = 5$ is a horizontal asymptote, exemplifying the limit $\lim_{x \to 2} f(x) = 5$. Two other students considered a vertical asymptote at $x = 2$ (Figure 4.4).

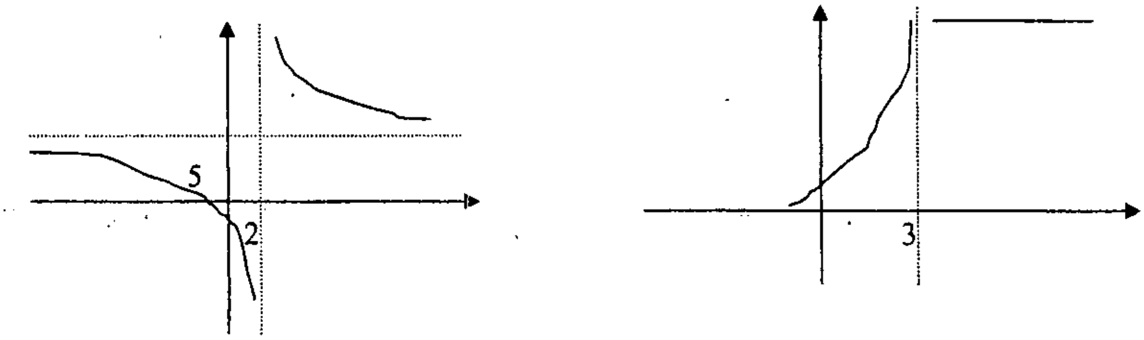


Figure 4.4 More student's sketches of $\lim_{x \to 2} f(x) = 5$

Other students used the symbols 2,99... or 3^- to suggest that the function continually approaches an asymptote (Figure 4.5). In the last example we see evidence that the limit was perceived as a process, that approaches a fix value. As an interviewee stated, "*the function approaches 3 more and more, that is why I said that the limit is* 2,99...."

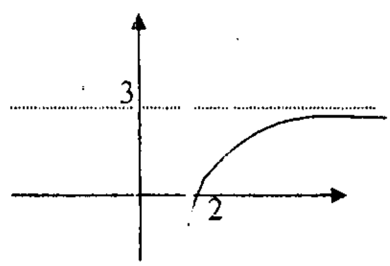


Figure 4.5 What is the limit of the function?

Nino held similar mental images for limits and asymptotes. Both being viewed as a value or a line that the function continually approached. His answer to a question about the role of an asymptote in the graph was, "An *asymptote of a graph is the* *maximum and the minimum value of a function (limit). Note that the function does not attaint the asymptote, it assumes very close values*". João's response to the same question also evidenced that the concepts 'limit' and 'asymptote' are interchangeable. He wrote, "*The role of an asymptote is to indicate the limit of a function*". In addition the concept image of a continuous movement suggested by the verbs *approaches*, *tends to* and *converges* misled the students to think that a limit demands a continual approach towards a point. This would be an unfinished process and only possible if there was an asymptote at that point. Therefore, if x is continually approaching 2, $y = 5$ might be an asymptote.

I suppose that the image of the limit as an asymptote might also, to some extent, be attributed to the students' practice in or outside the classrooms. I observed in some teachers' worksheets and the national examination from 1976 to 1998 that in each question where a graph was given to indicate the limit, there were asymptotes or discontinuities at those points. In these sheets or examination, the students were never asked to indicate the limit of a continuous function. Williams (1991) encountered this (limit as an asymptote) understanding of limits in one college student studying calculus. The numerical and graphical representations provided a similar concept image, because both had a motion nature. However they were different, because while in the previous category the function approached infinity, the *motion picture* category included limits where both variables approached finite values.

4.3.1.2 Motion picture: as x gets to a the corresponding values of the functions approaches b indefinitely.

Five students held a *motion picture* concept image of the concept of a limit of a function (Table 4.2). For instance, some of them used the term *approaches to* or *tends to*; to answer the question "what happens with y when x approximates to 1,5?" in the numerical representation of Question 4.2. To justify his limit result of the function represented in Figure 4.6 in the interview, Pedro affirmed that, "*The limit is 3 because the values of the function approach more and more and attain the limit 3*". This demonstrated that the student held a concept image of the limit as a movement towards a point.

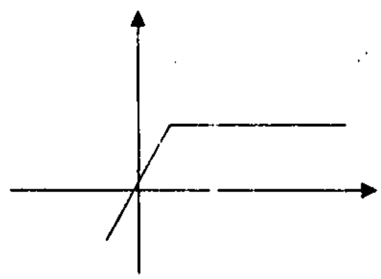


Figure 4.6 What is the limit of the function?

Most of the time, limits of constant functions as x tends to infinity (Figure 4.6) or functions of the type $\lim_{x \to a} f(x) = b$ being a and b real numbers represented graphically, provided contradictory responses. A few students (7%) regarded them as an approach to a value, and used the verbs *tends* to and *approaches to* to infer a sense of motion or movement. These students were classified as having a *motion picture* concept image. Alternatively, other students rejected the dynamic image of limits when working with a limit to infinity of constant functions or a limit of a continuous function at a point, rather they associated it with a static image, *value correspondence*. Besides the *value correspondence* concept image, the students developed two more static images, *barrier* and procedural images.

4.3.1.3 Value correspondence: the limit is a y value corresponding to a x value. Normally it will not be attained but occasionally it may. The symbol = sometimes means equal to and other times very close.

Some students' papers led me to think that they considered the limit as a value of a function. According to the students, the limit of a constant function might be assumed. Alternately, the limit of an increasing or decreasing function with a horizontal asymptote is not attained. It appears that in the first situation they evoked a mental image of a constant function as a constant value of a function for all x values. The limit of this function is a constant because it belongs to the range of the function. Differently, the value of the horizontal asymptote is not reached. Therefore it is not the limit because it does not belong to the range. For instance, in Question 6.3(b), the students were required to indicate the limit of the function with a horizontal asymptote $y = 3$. Some students wrote that the limit is 2,99... to show that the limit was a value of the function. Similarly, Ben and Bertino stated in the interview, "*the function tends to 3*". Alternatively, in Question 6.3(d) given a graph of a constant function Bertino replied, " *I think that here the function does not tend to, but it can reach the value 3. It does not tend to 3 but it is equal to 3*". In addition, Aldino explained that, "*the limit tends to 3 because the graph does not attain the limit 3, it tends to, approaches the limit 3 and never touches it. While in the last case, I said that the limit is 3, because the graph ends exactly in the value 3*". For these students the function in Question 6.3(b) dynamically approaches the limit but does not reach the value 3, while in Question 6.3(d) they did not consider that the function also approaches the limit. Their sentences stressing that the function did not tend to, but simply equals 3, showed once more that the students associated the words 'tends to' to motion. Accordingly, these students held the *correspondence value* concept image. This means that the limit is a value that the function must assume. In the case of a horizontal asymptote, the limit might be a closest value to the value of the asymptote that the function attained.

4.3.1.4 Barrier: the limit is greater or lower than all values of the function

The Portuguese term "limite" (limit) most frequently brings to mind a boundary point or a boundary line. For the students, mathematically this boundary image was a value, which can be greater than or less than all values of the function. According to their sketches the limit is never attained. This distinguishes this category from value correspondence (Section 4.3.1.3)

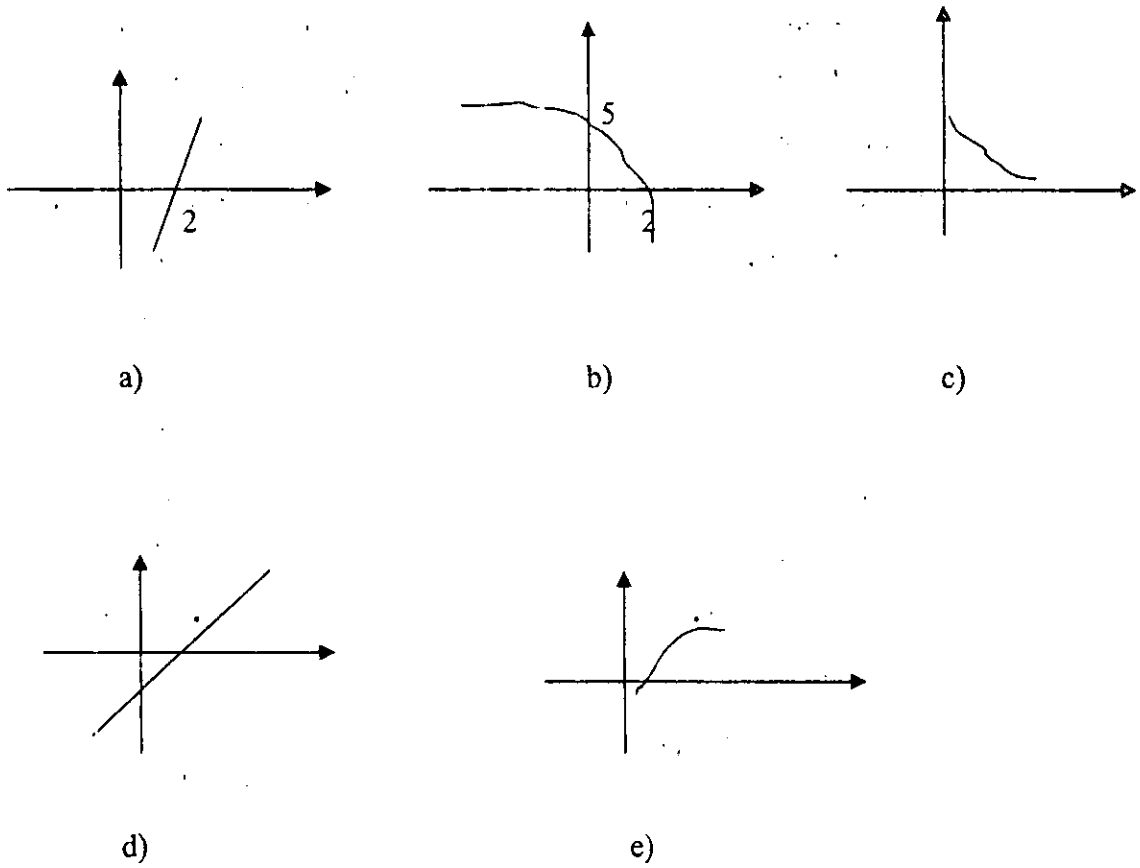


Figure 4. 7 Sketches of $\lim_{x \to 2} f(x) = 5$

As we see, the students attempted to draw graphs where all values of the function stayed within the limit of the function. The function in the first graph assumed the value of the limit, but all values of the function stayed within this limit (Figure 4.7(a)). It means that the barrier was attained. In all other cases the limit is not attained. Although there was no asymptotes in Figure 4.7 (b), the sketch suggested imaginary asymptotes (both horizontal and vertical) as barriers. Alternatively, it appears that in Figure 4.7 (c) the student thought of an interval that limited the function. In Figure 4.7(d) the student traced a straight line and marked the point (2,5) off but very close to the line, to show that it is a barrier. However, due to other shortcomings, namely a failure in comparing the coordinates of two points, he did not notice that there were some points of the function greater than 5. Gomes who drew the graph in Figure 4.7 (e) stated in the interview that the point (2,5) did not belong to the graph because 5 was a barrier and the function did not reach it. In the interview Pedro, also holding a *barrier* concept image rejected his answer to Question 6.3(c). He justified his rejection saying that zero could not be the limit because some values of the function were greater and some smaller than the zero (Figure 4.8).

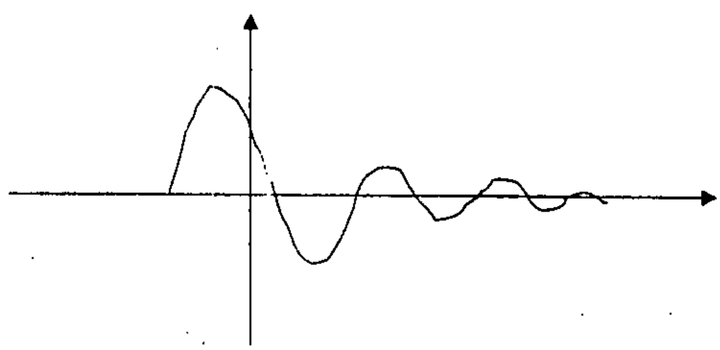


Figure 4.8 Question 6.3(c) What is the limit of the represented function when x gets bigger?

4.3.1.5 Procedural: the limit concept is encapsulated in the mathematical procedure and rules.

The task revealed that the students frequently associated the limit to algebraic manipulation. Their concept image of a limit as a procedural method was substantiated by the students' need of analytic functions where rules, formulas and procedures were applied. The students work in the different representations provided evidence of the dominance of the *procedural* concept image. In the task, most of the students resort to algebraic calculations to respond to questions given in the numerical or graphical representations (Questions 4.2 and 6.4). For instance, they attempted to get a formula of pairs of values to indicate the limit or to complete tables of values when they were given a graph.

Concept images may be discerned as one works in the different representations. Therefore the analyses of the tasks that associate the representations allowed me to compare the representation form with the concept images. The student's work in the different representations and the concept images produced by the strategies used to solve the task will be focused on in the next section.

4.3.2 THE REPRESENTATION SYSTEMS

4.3.2.1 NUMERICAL REPRESENTATION

The task contained two questions using a numerical representation. Firstly, Question 4.2, where the students were asked to indicate a limit given a table of values of a linear function. Secondly, Question 6.4 where the students were asked to complete a table of values of a function given its graph. The students used either a formula or looked for the behaviour of the function to give the limit. However, for a considerable number of answers, 43,7% in Question 4.2(a) and 49,3% in Question 4.2(b), I was not able to determine which strategy the students used because they merely gave the answer without any calculation or explanation. The summary of the students' strategies in the question is presented in Table 4.4. The table shows that the majority of the students (50,7%) did not attempt to complete a numerical table from a given graph (Question 6.4), while 9,8% were not able to indicate the limit in the table in Question 4.2(a). In addition, only 14 (19,7%) students correctly responded to Question 6.4 There were also 33 (46.5%) and 37 (52,1%) correct answers to Questions 4.2(a) and 4.2(b) respectively. It appears that the numerical representation caused lots of difficulties for the students.

Table 4.4 Students' strategies used to respond the numerical Questions 4.2 and 6.4

Students' strategies	Q 4.2 (a)	Q 4.2 (b)	Q 6.4
No answer	7 (9,8%)	12 (16,9%)	36 (50,7%)
Formula	-	24 (33,8%)	14 (19,7%)
Look for the behaviour of the points on the table or graph	33 (46,5%)	-	21 (29,6%)
No explicit strategy can be seen in the written work	31 (43,7%)	35 (49,3%)	-

Responses to Question 4.2

The students were asked to determine a limit of a linear function represented by a table of values. Less than half of the students correctly answered the two sub-questions. To solve sub-question 4.2(a) some students looked for the behaviour of the function and indicated either a value or the behaviour of the function. Thus, they answered that the function increased or decreased. None of the students considered the two-sided behaviour of the function at the given point. Pedro declared in the interview that he got the value 0,5 by seeing what happened with the y values as the x values increased. He was not concerned about the behaviour of the function when x decreased. Alternatively, for sub-question 4.2(b) 33,8% of the students did not read the result directly from the table (Table 4.5). They shifted from the numerical to the algebraic representation before indicating the limit. Accordingly, they tried to find a formula on the basis of the given pairs of values. The obtained limit did not fit with the values of the table in some of the cases. For example, some students wrote:

$$\lim_{x \to 2} f(x) = f(2) = 2$$

$$\lim_{x \to 2} f(x) \Leftrightarrow \lim_{x \to 2} f(2) \Leftrightarrow \lim_{x \to 2} 0.2 = 0.$$

Alternatively, three other students wrote the limit correctly, even though they were different from the limits of the chosen analytic expressions. Examples of the students' inferred formulas were: $y = 1{,}4x$ and $y = 2x$. The limits of the supposed functions were different from what they wrote as the answer to the question. It seems that they held the *procedural* concept image, but they probably noticed that the limit of those expressions did not fit in the table. Consequently, they decided to keep both responses. Likewise, in the interviews more then half of the interviewees (62,5%) attempted to find an expression or regularity and then evaluated the limit to answer this question. The remainder based their approach on the dynamic nature of the points to find the limits as following protocols showed,

> Arthur: "I found this value because I saw that as x comes closer and closer to 1,5 the y values approach 0,5" (pointing the left side of 1,5).

> Bertino: "ok, here to evaluate the limit of the function, firstly I had to have an expression of the function. It means I had to determine the value of the slope. I substituted the values of the coordinates in the general expression $y = ax + b$ and determined the coefficients a and b. After having the expression of this function I evaluated its limit"

The first protocol suggested the *motion picture* concept image, where the approach of the x values towards 1,5 implies the move of the y values towards 0,5, therefore a dynamic concept image. The second example showed the students' inability to indicate the limit in the numerical representation. He resorted to an expression to work on, showing in that way the ability of translating from the numerical to the algebraic representation, since he felt more confident evaluating limits algebraically. This act evidenced the predominance of the *procedural* concept image. Consequently, the numerical representation as a way of determining limits was avoided by the excessive use of the *procedural* concept image.

Responses to Question 6.4

In Question 6.4 the students were requested to complete a table of a given graph as accurately as possible.

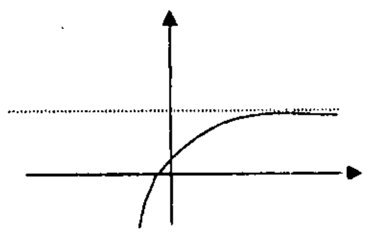


Figure 4.9 Graph given to complete a table

A few students (18,6%) completed the table correctly. They might reasonably evaluate the values of the function evoking their *motion picture* image. Here again, as Table 4.4 portrays, another (19,7%) students attempted to find an analytic function to help them to complete the table. For example $y = 3x$, $y = b^x$ or $y = \frac{\log x}{x-3}$. As a result the deduced values were greater than its limit, however those students did not notice this. In the interviews similar strategies were developed to fill the table. Actually, half of the students tried to find an expression or regularity from the values of the table as evidenced in the transcript of interviews with Bertino and Humberto. It seemed that the *procedural* image of the limit concept was strong in these students.

I: "which strategy did you use to complete this table?"

B: "ahh, to complete the table? My analysis was that when x is 1 the y is equal to zero. Then I saw that it differs from one unit. Then if x is 1 y must be 2, when x is equal to 100 y is 99, when x is 000 y is 999 and so on".

I: "explain how you were complete the table?"

H: "my idea was the following, I do not know if it is correct. With these two columns I did not know how to do, but from here I used the following principle.

We had 10, so 10 squares is 100, then we have 10000, I see that it could be 10 power 4, and here could be 10 power 6".

Despite Bertino's incorrect answer, there was no contradiction between his answer and the limit he indicated for the function represented in the graph. His answer about the limit of the function was:

"The limit of the function is infinity, because as much as we approach the infinity, we do not arrive anywhere". (He pointed the x-axis)

For Bertino the limit of the function represented in the graph was infinity, what he meant by "not arriving anywhere", was:

"Here this function began in minus infinity, it has a horizontal asymptote in $y = 3$, it does not pass through 3, but it goes to the infinity"

This failure was most likely due to his confusion of the coordinates of a point. When Bertino said, "As we approach infinity" he was thinking of the x values, but the conclusion was erroneously made in relation to the x-axis. Similarly, Humberto considered the y values to indicate where the function began and where it had an asymptote. He stressed that the function did not pass through this value. Nevertheless, he added that the function went to infinity. In the last statement he was working with the x values.

The students' statements exemplified that the numerical table and graphical task did not stimulate the students' *motion picture* concept image rather they needed an expression to solve the task. This means that by relying on the algebraic calculations, they were invoking a *procedural* image to answer the question. In addition, Furina (1994) outlined that the students consider the results of algebraic calculations as "real" or firm, whilst numerical results are considered as being partial or tentative evidence.

Forty-six out of seventy one students left the numerical task blank. When the students worked on a numerical representation, they usually substituted a x value to obtain the y value using mathematical rules and procedures, also an algebraic calculation. In addition if they have to indicate a limit of a function on the basis of a numerical table, they tried to find an expression and from it evaluate the limit using the different methods. It seems that they felt more comfortable working with the algebraic representation.

4.3.2.2 ALGEBRAIC REPRESENTATION

In this section I will analyse the students' answers to the algebraic tasks looking for the used strategies. In addition I am interested in the concept images that directed their algebraic manipulations. The student's concept image of an algebraic calculation appears to have become a standard procedure for evaluating limits. This procedure involves formulas, methods, rules that must be applied successively no matter irrespective of an indeterminate form. Accordingly, I classified this conception as a *procedural* concept image. In the algebraic calculations (Question 7.1), with exception of the two last exercises, the students attempted to compute all other limits, despite their difficulties with the mathematical rules, the methods and the indeterminate forms. The different methods and the mathematical rules and procedures of evaluating limits caused them difficulties. However, the major problem lay in the disassociation between the mechanical calculations and the limit concept.

4.3.2.2.1 Students' algebraic manipulations

Although the algebraic manipulation is not central to my research I have chosen for completeness to include a brief discussion of the students' manipulation strategies.

Factorization

The students were asked to evaluate the limit $\lim_{x \to +\infty}(x^2 - 3x - 4)$. Artur was asked in the interview to justify his answer. He said.

> A: "I replaced the infinity and I obtained $\infty - \infty - 4 = \infty$ "
>
> I: "but, is $\infty - \infty$ not an indeterminate form?"
>
> A: "no, because the first term is squared, is bigger than the second one, which is three times infinity. So the limit result is plus infinity"

Some students had difficulty imagining the difference between the infinities resulting from the substitution of infinity in the terms of the polynomial. Their concept image of infinity is a very big number with no degrees. Thus, they considered the difference as an indeterminate form. To evaluate the limit they used different strategies. Examples thereof are:

$$\lim_{x \to +\infty}(x^2 - 3x - 4) = \lim_{x \to +\infty}[(x-4)(x+1)] = +\infty$$

$$\lim_{x \to +\infty}(x^2 - 3x - 4) = \lim_{x \to +\infty}\left[x^2\left(1 - \frac{3}{x} - \frac{4}{x^2}\right)\right] = \lim_{x \to +\infty} x^2 = +\infty$$

Some students employed factorization to evaluate a limit in which the indeterminate form was $\left[\frac{0}{0}\right]$. I was not able to determine if they substituted the variable x with infinity because only a few students wrote the indeterminate form. In my lessons, I observed students who were not used to writing the indeterminate forms or even resolving them. Alternatively, some students did it mentally (eg Gomes) or in rough copy (eg Pedro). In the interviews they explained,

I: "why did you factorize her?"

G: "I didn't write but first I did the substitution and I got the indeterminate form $\frac{0}{0}$, which requires a factorization"

P: "here we get an indeterminate of the type $\frac{0}{0}$, therefore I factorized the terms of the fraction"

I: "how do you say there is an indeterminate?"

P: "I substituted in the rough copy"

An example of students' factorization is

$$\lim_{x\to 2}\frac{x^2-5x+6}{x^2-4}=\lim_{x\to 2}\frac{(x-3)(x-2)}{(x-2)(x+2)}=\lim_{x\to 2}\frac{(x-3)}{(x+2)}=-\frac{1}{4}.$$

The image of a factor that is annulled by the value 2, and therefore cancelled, directed them to factorize the terms of the function. Some students' incorrect factorizations were:

$$x^2-5x+6=x(x-5)+6$$

$$x^2-5x+6=(x+3)(x-2))$$

Division of the terms of a fraction by the same expression

A few students divided the numerator and the denominator of the fraction by the same expression. It seemed that these students did not substitute the values 3 into the function. Illustrations of the students' calculations are:

$$\lim_{x\to 3^+}\frac{x+2}{x-3}=\lim_{x\to 3^+}\frac{\frac{2}{x}+\frac{x}{x}}{\frac{x}{x}-\frac{3}{x}}=\frac{\frac{2}{3}+1}{1-\frac{3}{3}}=\frac{5}{0}=+\infty,$$

$$\lim_{x\to 3^+}\frac{x+2}{x-3}=\lim_{x\to 3^+}\frac{x-3+5}{x-3}=\lim_{x\to 3^+}1+\frac{5}{x-3}=1+\frac{5}{0}=1+\infty=+\infty$$

In the penultimate step of each evaluation the students obtained the same results as when they substitute immediately the values in the functions. This was the reason that leads me to think that they did not substitute the value before. Students' common errors was:

$$\lim_{x\to 2}\frac{2-x}{x-3}=\lim_{x\to 2}\frac{\frac{2}{x}-1}{1-\frac{3}{x}}=-1$$

They conceived of the image that the quotient $\frac{k}{x}$ was always zero independent of the approaching process.

L' Hospital rule

Three students employed L' Hospital rule to evaluate the limits with the indeterminations $\left[\frac{0}{0}\right]$ and $\left[\frac{\infty}{\infty}\right]$ as shown in the example.

$$\lim_{x\to 2}\frac{x^2-5x+6}{x^2-4}=\lim_{x\to 2}\frac{(x^2-5x+6)'}{(x^2-4)'}=\lim_{x\to 2}\frac{2x-5}{2x}=-\frac{1}{4}$$

Application of the conjugate

The factor $x-9$ is divisible by the term $\sqrt{x}-3$. Therefore, there exits an expression that multiplied by $\sqrt{x}-3$ is equal to $x-9$, namely $\sqrt{x}+3$, which is the conjugate of $\sqrt{x}-3$. The students multiplied both terms of the fraction by $\sqrt{x}+3$ as the example portrays

$$\lim_{x\to 9}\frac{x-9}{\sqrt{x}-3}=\lim_{x\to 9}\frac{(x-9)(\sqrt{x}+3)}{(\sqrt{x}-3)(\sqrt{x}+3)}=\lim_{x\to 9}\frac{(x-9)(\sqrt{x}+3)}{(x-9)}=6$$

Substitution of the variable

With the aim of eliminating the radical, some students substitute the variable x by another variable. This procedure implies the change of the value to which the function tends. A student's evaluation of a limit applying the substitution method is presented below.

$$\lim_{x\to 9}\frac{x-9}{\sqrt{x}-3}=\left|\begin{matrix}x=t^2\\ t\to 3\end{matrix}\right|=\lim_{t\to 3}\frac{t^2-9}{t-3}=\lim_{t\to 3}\frac{(t-3)(t+3)}{(t-3)}=6$$

A common mistake was to keep the new variable t tending to 9.

The identity $\lim_{x\to +\infty}\left(1+\frac{1}{x}\right)^x=e$

Some limits, with indeterminate form $[1^\infty]$, were evaluated with the help of the identity $\lim_{x\to +\infty}\left(1+\frac{1}{x}\right)^x=e$. An example of a student' s work is:

$$\lim_{x\to +\infty}\left(1+\frac{1}{x}\right)^{3x}=\lim_{x\to +\infty}\left[\left(1+\frac{1}{x}\right)^x\right]^3=e^3$$

_e_ number formula

A formula, without any explanation about its origin or deduction was taught, to evaluate the limit $\lim_{x\to+\infty}(f(x))^{g(x)}$ with the indeterminate form $[1^{\infty}]$. To calculate this limit the formula $\lim_{x\to+\infty}(f(x))^{g(x)} = e^{\lim_{x\to+\infty}[(f(x)-1)g(x)]}$ was employed. An example thereof is:

$$\lim_{x\to+\infty}\left(\frac{x+3}{2+x}\right)^{x} = \lim_{x\to+\infty}\left(\frac{x+2+1}{2+x}\right)^{x} = \lim_{x\to+\infty}\left(1+\frac{1}{2+x}\right)^{x} = e^{\lim_{x\to+\infty}\left[\left(1+\frac{1}{2+x}-1\right)x\right]} = e^{\lim_{x\to+\infty}\frac{x}{2+x}} = e$$

In the algebraic manipulation, handling zero or infinity as a term of a fraction is a challenge for the students. It seems that they rote-learned the results without thinking about the reasons behind the results. For instance, answers such as $\frac{a}{0}=a, \frac{a}{0}=0, \frac{0}{a}=0, \frac{0}{a}=a, \frac{a}{\infty}=a$ were common in the task. In addition, almost all the students did not use the sign plus or minus before the infinity ($+\infty$ or $-\infty$) symbols, when they calculated the limit of a quotient whose numerator was a real number and denominator tends to zero. They did not take into account the sign of the denominator, which is actually a positive or negative number very close to zero and not equal to zero. Their algebraic calculations, the substitution of the x value into the function and the successive applying of rules and procedures, were divorced from the limiting process. Only eighteen students, half from each school, used the positive or negative signs but those who were interviewed gave an incorrect justification. One of the students', Gomes, choice of the sign was determined by the numerator.

I: "you have this result, infinity. Is it just infinity"

G: "no, it is plus infinity"

I: "why is it positive?"

G: "because the numerator is positive"

I: "and if we had a negative numerator?"

G: "it would be minus infinity"

Alternatively, for Pedro it depended on the side from which the limit approached.

I: "what was your result for $\lim_{x \to 3^+} \frac{2-x}{x-3}$?"

P: "I obtained infinity"

I: "just infinity?"

P: "......."

I: "infinity, without any sign?"

P: "ah! It is positive"

I: "why positive?"

P: "because the variable x tends to 3 on the right"

I: "and what would happen if the x tended to 3 on the left?"

P: "the sign would be negative"

Despite the difficulties in applying the methods and in mastering the algebraic rules and procedures the students attempted to evaluate the limits. The major problem was in understanding the meaning of the obtained results. For them the limit result was a number, which had nothing to do with the limit concept. The following extract of an interview elucidates this claim.

I: "which difficulties did you face with the limit concept?"

H: "this was the most difficult topic in Grade 12. It appeared to be simple but it requires greater comprehension".

I: "and would you explain the problems you faced?"

H: "the problem I see, I think we had to have a worksheet with the methods and formulas to evaluate the limit, so that we can look and see which method or formula we had to apply in each case and then evaluate the limit".

I: "ok, but if you had the method and evaluated the limit correctly, for instance if you had this limit result, do you know what this result means?"

H: "yes, that I know, if I have for instance 2 as a result it means that when it tends to 3 on the right or on the left it will always be equal to 2"

I: "and in this case?"

H: "in this case the limit of the function when it tends to plus infinity will be equal to plus infinity".

The student simply read the limit without any understanding of its meaning. On the other hand, he argued that the problems he encountered concerning the limit concept were only related to algebraic calculations and regretted that a worksheet that guide the students to solve the limit tasks algebraically did not exist. Thus, he encapsulated the limit concept in the algebraic calculations. This is evidence of the dominant *procedural* concept image among the students. However, when the students were asked to change from the algebraic representation to the graphical one, other images such as *asymptotic* or *value correspondence* (Question 6.7(a) and 6.7(b)) were evoked.

Shifting from the algebraic representation to the graphical representation the students were required to deal with new aspects that belonged to this representation, for instance, the variation of the coordinates when a point moves along the curve. The students must use new strategies to deal with these aspects, and it consequently enhances the development of their concept formation. In addition the algebraic limit results may be sustained by the visualization of a graph. The next section will give an overview of how the students elaborated the task within the graphical representation.

4.3.2.3 GRAPHICAL REPRESENTATION

Using the graphical representation for limits allows students to illustrate their understanding of the meaning of the limit results and the implication of the symbol $x \to a$. Secondly, analytic results may be supported graphically. In spite of their confidence working with the algebraic representation, the change to the graphical and numerical representations challenged the students, and sometimes produced contradictory responses for the same attribute. For instance, Pedro working in the algebraic representation, when asked to explain what he meant by $\lim_{x \to 4} f(x) = 1$, replied that, "*When x approximates to 4, y approximates to 1, it cannot be equal to 1*". On the other hand, he stated that the limit of the function, in the graph represented in Figure 4.10, when x gets to infinity is equal to 3.

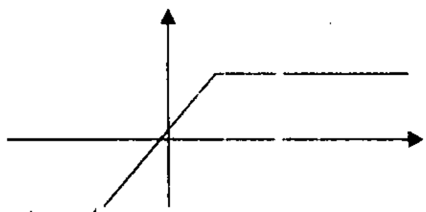


Figure 4.10 A graph to indicate the limit

Accordingly, he obtained different responses to similar questions. The limit might not be attained in the algebraic representation, while it might be attainable in the graphical one. The student might recall the limit through an *asymptotic* concept image, which cannot be reached in the first case and as a *value correspondence* image in the second one. This is an indication of the compartmentalization of the students' knowledge. Vinner and Dreyfus (1989) assigned to Vinner, Hershkowitz and Bruckheimer the introduction of the compartmentalization phenomenon, a phenomenon that "Occurs when a person has two different, potentially conflicting schemes in her or his cognitive structure" (p. 357). He added that besides the compartmentalization another reason for the inconsistency might be the fact that a given situation did not stimulate the most relevant scheme for the situation under consideration. Therefore, irrelevant schemes for the given situation are evoked. The compartmentalization of the representations might be the cause of students' difficulties in transferring an algebraic result of a limit of a function to its graph. A considerable number (65,3%) of students did not answer Question 7.3. Only six students, two from Maputo and four from Quelimane, sketched the graph correctly.

Among them all but three used other information such as the zeros of the function, the ordinates in the origin, but did not use the result of $\lim_{x\to 3^+}\frac{x+2}{x-3}$ as was asked in the question. Humberto, who sketched the graph correctly said in the interview,

> "So, to sketch the graph I thought that. as I have x tending to 3, 3 is the x value. The limit is infinity that means the y gets to infinity. So this is possible only in the case there is an asymptote"

On the other hand, Pedro stated,

> "As I have here, when x tends to a real number, 3 in this case, the limit is infinity, I have an asymptote, because as closer as the x approaches 3 the function approaches infinity"

Both students demonstrated a consistent behaviour between the algebraic and the graphical representations. They elaborated the task recalling suitable concept images. Alternatively, the majority (76,9%) of the students that tried and did not accomplish the task tried to plot the points obtained in the table completed in Question 7.2, but they could not draw the graph because of difficulties they encountered with the symbol $+\infty$ matching the x value to 3 (see Figure 4.11). They might have imagined that $x = 3$ did not belong to the domain, that is why they interrupted the curve in the neighborhood of $x = 3$, but they did not associate this fact with a vertical asymptote. Therefore, they held an incomplete *asymptotic* concept image.

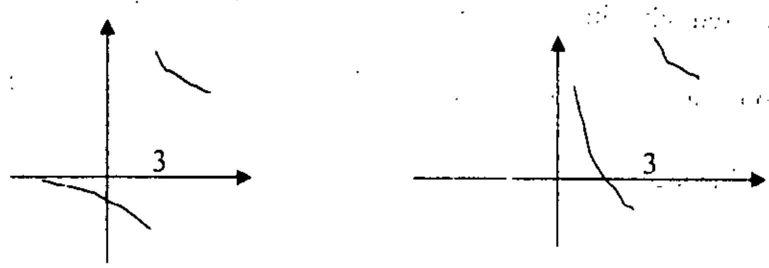


Figure 4.11 Students sketches of $\lim_{x \to 3^+} f(x) = +\infty$

Pedro, trying to overcome the obstacle he encountered with $+\infty$, argued, "*The graph of a sequence is defined through points*". With this justification he plotted the points without joining them to obtain a curve. He was aware of the inconsistency between joining points and the limit being infinity as x tends to 3. In the same way, the majority of the students failed to graph an algebraic result or a numerical table containing infinity. Rather, the students were used to calculating limits in the study of characteristic points of the function, but they did not use those results when they drew the graph. To elucidate, very few students attempted to solve Questions 6.7 and 7.3, and among them only a small number accomplished them correctly. The graph in Figure 4.12 showed a contradiction with the resultant limit.

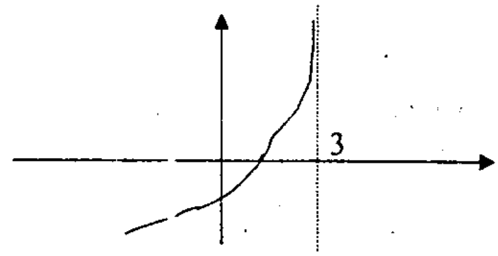
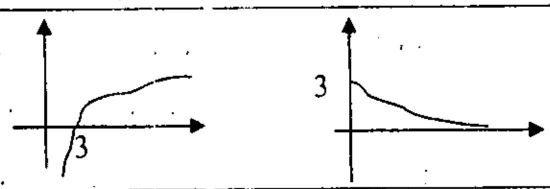


Figure 4.12 Figo's sketch of $\lim_{x \to 3^+} f(x) = +\infty$

Figo's algebraically determined limit was positive infinity. He may have known that when the variable x gets to a real number and the resulting limit was infinity there was a vertical asymptote at that point. He drew the vertical asymptote properly and sketched the graph approaching the asymptote. However, he drew the graph on the left of the asymptote, instead of drawing it on the right as the task asked. This student' s concept image of the limit was *asymptotic*, but he did not cope with the lateral approach of the graph to the asymptote. The students' patterns of sketching graphs given limits of functions are summarized in Table 4.5.

<u>Table 4.5</u> Graph sketching incidents and possible reasons for the students' errors.

	Question	Incidents	N. of students
1	$\lim_{x\to+\infty} f(x) = 3$	Horizontal asymptote	17
		Constant function	6
		$f(3) = 0$ and y increasing to infinite positive	5
			4
		No answer	39
2	$\lim_{x\to 2} f(x) = 5$	$f(0) = 5$ **and** $f(2) = 0$	5
		curve passing closer to the point (2,5)	1
		vertical asymptote: $x = 2$; horizontal asymptote: $y = 5$	5
		$f(2) = 5$ is a point of the curve	13
		$f(2) = 0$ and horizontal asymptote: $y = 5$	4
		No answer	43

	Question	Incidents	N. of students
3	$\lim_{x \to 1^+} f(x) = -\infty$	Vertical asymptote	7
		$f(1) = 0$ and y decreasing to minus infinite	10
		Horizontal asymptote	1
		$x \to -\infty$	6
		No answer	47
4	$\lim_{x \to +\infty} f(x) = +\infty$	$\lim_{x \to 0^+} f(x) = +\infty$ and $\lim_{x \to +\infty} f(x) = 0$	4
		y $\to +\infty$ when $x \to +\infty$	11
		No answer	56

The table shows that a considerable number of students were not able to graph given information. The high number of students who gave no answer demonstrated a general inability to translate from the algebraic representation to the graphical one. In the first example, $\lim_{x \to +\infty} f(x) = 3$, 16 students recognized either the horizontal asymptote or the constant function in infinity. Their sketches suggested that they understood the symbol $x \to +\infty$ and they mastered the changes of the coordinates when a point of the graph moved. Alternatively, five students considered $f(3) = 0$ and made the y increase to infinity. They did not notice that there existed a horizontal asymptote at this point and they changed the coordinates. In addition, two students also drew the graph as it was shown in the table.

In the second example $\lim_{x \to 2} f(x) = 5$, thirteen students regarded a limit as an asymptote. Consequently, they sketched asymptotes, either horizontal $y = 5$, or vertical $x = 2$, or both asymptotes. Alternatively, other students considered the point (2,5) not belonging to the curve and plotted the point and drew the curve not on this point (see Figure 4.13)

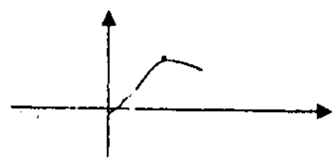


Figure 4.13 Illustration of Alda's graph holding a *barrier* concept image.

It is noticeable how the point in Figure 4.13 forms an upper barrier for the graph. These students held a *barrier* concept image where the function may not assume the limit or even pass the barrier.

Thirdly, the limit $\lim_{x \to 1^+} f(x) = -\infty$ caused problems for the students. Four students having an *asymptotic* concept image thought of the x as 1 and because of that they drew the curve through the point (1,0). Then, they thought of the y variable getting to negative infinity and sketched the graph as represented in Figure 4.14.

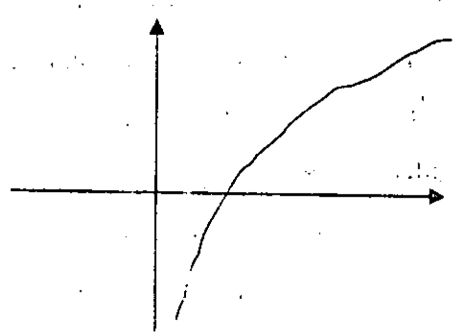


Figure 4.14 A student's graph of $\lim_{x \to 1^+} f(x) = -\infty$

In the last example, students found sketching the graph of the limit $\lim_{x \to +\infty} f(x) = +\infty$ difficult. Only eleven students attempted to draw the graph, and no more than seven did so suitably. Some students linked the sequence of each variable with a different sequence that was not given. For instance, they noticed that the variable x might approach positive infinity, but they allied it with a sequence of y values that gets closer to zero, and not to positive infinity as was required. They made the same mistake concerning the *y*, which tended to infinity, and was associated with a sequence of *x* values approaching zero. It seems that these students conceived of an *asymptotic* image of a limit of a function. Namely, as a vertical asymptote when the function tends to infinity, on one hand, and as a horizontal asymptote when *x* tends to infinity on another hand. There was also no indication of real values in the limit notation to write the asymptotic expressions. Therefore, they sketched the coordinates axes as asymptotes.

In summary, each of the five concept images identified was associated with the graphical representation. Alternatively, the numerical representation appeared to be linked to the *motion picture* and the *procedural* image while the algebraic representation was only related to the *procedural* image. In addition, the students felt reasonably comfortable working with formulas. A considerable number of them tried to get a formula or an expression from a graph or a table when asked to determine a limit given these two representations. Then they answered the question starting from the obtained formula or expression. Examples can be seen in Chapter 4.3.2.1 and 4.3.2.2 where the students tried to find an analytic exponential function of the represented function to complete the table. However, they sometimes got contradictory answers, when changing from one representation to another. The inconsistency of their responses was either due to the compartmentalisation of the representations or because the circumstance did not prompt suitable images and instead less applicable images were evoked. The link of the algebraic calculations or graphics to a context problem produced similar results. The mental representation the students held of a physical object being modeled was very often incompatible with the images generated from the mathematical model itself. Dreyfus (1991) distinguished images generated from a mathematical model and those that are generated by the physical object being modeled. Including a task on modeling reality further allowed me to ascertain the consistency of students' images.

4.4 THE LIMIT IN APPLICATION TO A CONTEXTUAL PROBLEM

We frequently hear from students, "*we do not know how to apply the limit concept in daily life*" or "*I do not know why we must study the limit of a function, because I do not see where we can apply it*". Mozambican teachers are not used to employing examples of daily life, where students may solve them with the knowledge acquired at school. Despite subjects, such as Physics and Chemistry that apply the limit concept, links to other contexts are not referred to in most mathematics lessons on limits. This connotes Mathematics as a body of knowledge that has nothing to do with reality. Boaler (1993) claimed, "the use of examples which students may analyse and interpret, it is suggested, allows students to become involved with the mathematics and to brake down their perceptions of a remote body of knowledge" (p. 118). The benefits of a link between the concept and daily life goes beyond Boaler's claim in that the link of a concept with an application or as a tool to solve tasks helps the evolution of the students' concept. Accordingly, I included in the task one question to see if the students can deal with a non-routine mathematical problem in a realistic situation.

In this task I wanted to focus on three aspects. Firstly, the students tried to answer the question "what happens to the temperature of a metallic bar removed from the fire if the temperature (T) (in degrees Celsius) in function of time (t) (in minutes) was expressed by $T(t) = 26 + \frac{90}{t+3}$" using mathematical reasoning. Most students thought mathematically with little reference to the context. They justified the temperature decrease saying that both magnitudes were inversely proportionally or, explaining that, as the magnitude time was in the denominator, the bigger the denominator the smaller the fraction.

Secondly, there were two questions with the same answer, one (5.2(c)) asking the approximate temperature of the room, and the other (5.2(d)) to calculate the limit of the given expression when t (time) tends to positive infinity. The students' answers to these questions are illustrated in Table 4.6.

Table 4.6 Students' responses to questions 5.2(c) and 5.2(d)

	Maputo	Quelimane
Both responses correct	2	7
5.2(c) wrong and 5.2(d) correct	6	5
5.2(c) correct and 5.2(d) wrong	-	5
5.2(c) left blank and 5.2(d) correct	11	4
$T=\infty$	6	13

Only two students from Maputo and seven from Quelimane gave a correct answer to Question 5.2(c). As we notice in the table above, many students calculated $t \to \infty$ correctly (Question 5.2(d)) but could not link it to the room temperature, either leaving the answer blank or giving an incorrect answer. For instance Gomes answered in the interview, "*I did not answer this question because the temperature of the room ...., I cannot say that it was zero degree Celsius. None of the data in the problem can confirm that, I mean nothing says that it was zero degree or other value*". Some gave no answer or tried to give a temperature that they thought of as a normal temperature on the basis of the mental representations of the physical object, the metallic bar. Illustrations of students' answers are the following:

"The temperature of the bar is that it had before it was put in the fire"
"It is that we feel in our environment, the normal"
"It was cool".

This illustrated that the students did not understand the meaning of the symbol $t \to \infty$ in the given context. They had the image of a metallic bar that they knew was getting cooler when removed from the fire. Thus, their concept images were evoked from the context of the problem and they did not associate these concept images with those acquired from the mathematics. Alternatively, five students gave an accurate temperature of the room, but could not calculate the value of the limit in Question 5.2d. It was not possible to determine what strategies they used to indicate the temperature because they only gave the temperature, without any explanation. However Arthur' s interview gave some insight.

> I: "what was the temperature in the room?"
>
> A: "26 degree Celsius"
>
> I: "why do you think so?"
>
> A: "because is the temperature of the bar at the instant the bar was removed. I consider this 26 of the given expression"

This protocol showed that the interviewee did not understand the structure of the expression and could not work with the associated algebraic procedures.

Thirdly, some students'claimed that the temperature increased showing that they did not relate the problem to the context in any way. They only imagined a mathematical expression as tool to solve their problem beyond any context. As Mohammad-Yusof and Tall (1996) reported, students think that mathematical problems consist of application of facts and procedures. Accordingly, this mathematical question was difficult to answer without using a mathematical justification. For them, the limit calculation resulted in a solution where the temperature of the metallic bar increased or remained constant over time. In Question 5.2(b) although, they answered that the temperature decreases they did not notice the contradiction in their mathematical calculation. They trusted their mathematical solution in Question 5.2(d), although this solution contradicted their contextual reasoning. Evidence of this can be seen in the responses of Edmundo and Susana.

5.2(a) What was the temperature at the moment the bar was removed from the fire?

Edmundo: $T = 56^0$

Susana: $T = 6{,}46^0$

5.2(b) What happened to the temperature of the bar as the time increases?

Edmundo: The temperature decreases

Susana: The temperature decreases

5.2(c) What was the room temperature?

Edmundo: $T = 56^0$

Susana: $T = 26^0$

5.2(d) What was the temperature of the bar after long time?

Edmundo: -

Susana: $T = +\infty$

Edmundo's solution of the initial temperature was equal to his room temperature. On the other hand, Susana's room temperature solution was higher than the initial temperature. However, both students answered in 5.2(b) that the temperature decreased. Susana calculated the limit when t gets to infinity and obtained infinity as a result. Likewise, nineteen students obtained the same result. This result is common for a limit evaluation, but it looks strange as a room temperature. However it did not sound strange for these students, because they might not associate the images of the context with the concept images of the mathematics involved. If they had, those images would be in conflict with those generated from the mathematics. Alternatively, Pedro demonstrated consistency between the mathematics concept image and the image provoked from the context, I: "why do you answered that the temperature of the room was 26^0?"

P: "the metallic bar was getting cooler, until it reached the temperature of the room. I think that the temperature of the room is that of normal time, even if the time is infinity the temperature of the room is already constant, therefore I tended the temperature to infinity".

In general, the Mozambican students are not used to solving problems in context. Therefore, when faced with a model in daily life they mismatched the concept image related to the context with those generated from the mathematics. In addition they managed to work out problems using mathematical facts or procedures, without logical reasoning and did not notice the contradiction between the answers. The contradiction might have been avoided if the results had been discussed keeping in mind the initial problem and not the mathematical model.

In this chapter the students' concept definitions of a limit of a function were split into two broad categories. The first category encompassed the definitions that included a dynamic nature. The dynamic nature was evidenced by the students' expressions such as *tends to*, *approaches to*, and *converges*. These verbs inferred a movement to a point of the function as the x values approached a fixed value. *Approach* and *continuity* were classified as dynamic concept definitions. Alternatively, the students' definitions indicated a static concept definition, using the words interval, neighborhood or the formal definition with the $\varepsilon - \delta$ symbols. As static concept definitions were considered the *notation/algebraic calculation*, *boundary* and *formal definition*. All three categories suggested a fixed value or distance. Similarly, the students' images were also categorized as dynamic or static in nature. The dynamic images included the *asymptotic* and the *motion picture*. Adding to these two categories three static concept images were classified: the *barrier*, the *value correspondence* and the *procedural* images. Moreover, I sought associations of the students' concept images and the different representation systems. Finally, I gave some insight into how the students linked the images associated to the context and those generated from the mathematics in a daily life problem.

CHAPTER 5

SUMMARY AND CONCLUSIONS

In Chapter 5.1 I elaborate two aspects in the process of solving tasks to complement the findings related to my research questions. Namely, the interaction between concept image and concept definition. and the interaction between concept images and the representations chosen to solve tasks. The findings of my research that are related to my research questions are summarized in Chapter 5.2. Chapter 5.3 describes the implications of the study for the investigators, the police makers and the teachers and Chapter 5.4 points out the limitations of the study. Finally, Chapter 5.5 offers some directions for future investigations.

5.1 CONCEPT DEFINITIONS, CONCEPT IMAGES AND REPRESENTATION SYSTEMS IN THE PROCESS OF SOLVING TASKS

Concept definition and concept image formation is essential for a complete understanding of a concept. When a student completes a task a definition or an image held may be evoked in the solution process. In addition, the solution of a task is only possible within a representation system. Consequently, in this section I want to give an overview of the students' consistencies or inconsistencies between concept definitions and concept images they held and the relationship between concept images and the representations used to solve tasks. These issues are not directly linked to my research questions, but it may be a contribution to further investigations about the students' actions when carrying out a task.

5.1.1 Interplay between students' concept definition and concept images and the way they performed the tasks

According to Tall and Vinner (1981) concept definitions are the specifications of a given concept using words, whilst concept images are all individual pictures one associates with the name of a concept. To solve a task one recalls a definition and/or images s/he relates to the concept under consideration. Vinner (1991) introduced four features of the process of problem solving or task performance drawing on concept image and concept definition. These are represented in Figures 5.1, 5.2, 5.3 and 5.4.

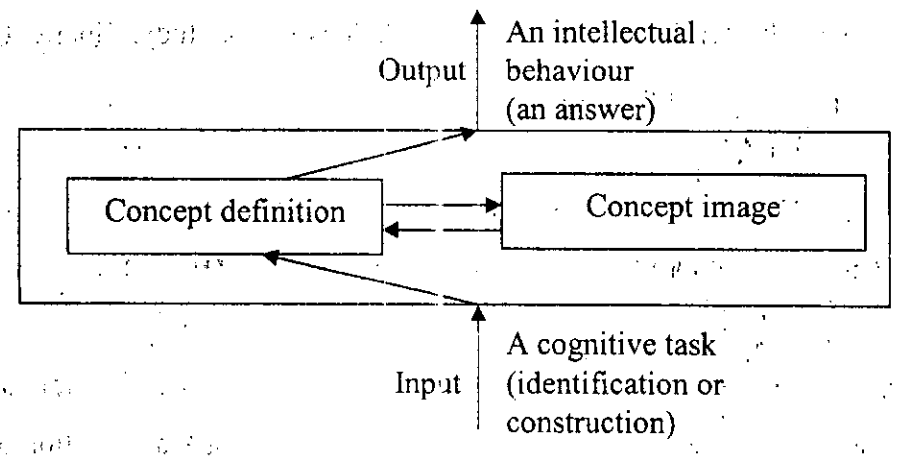


Figure 5.1 Interplay between definition and image

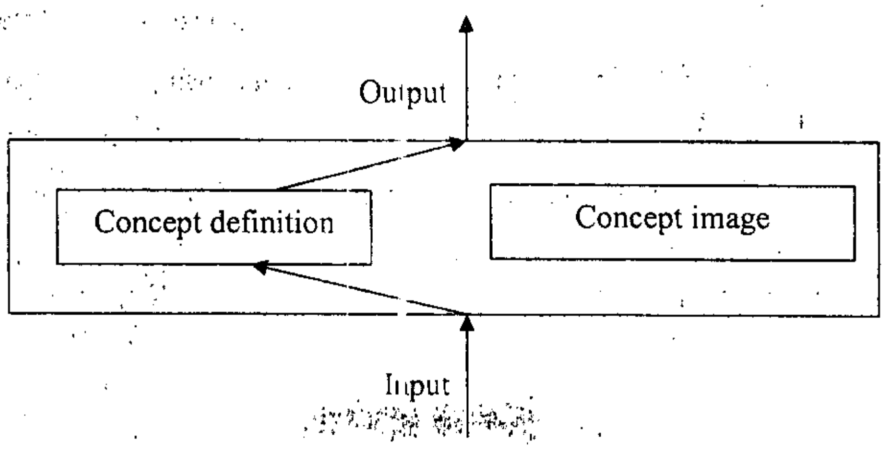


Figure 5.2 Purely formal deduction

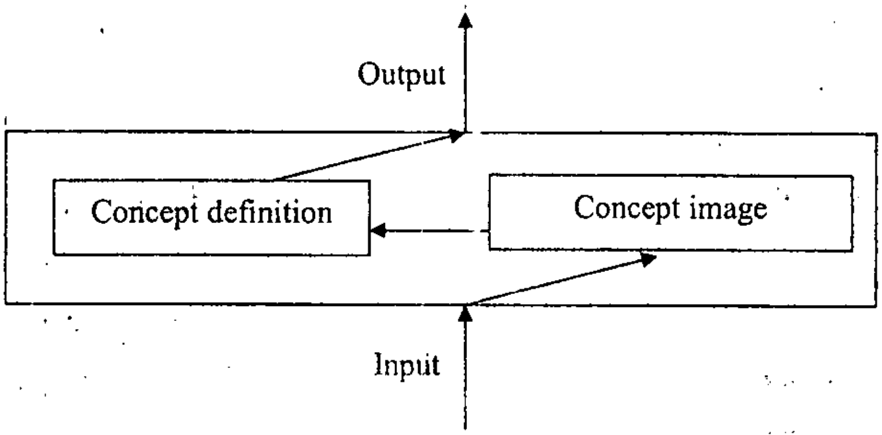


Figure 5.3 Deduction following intuitive thought

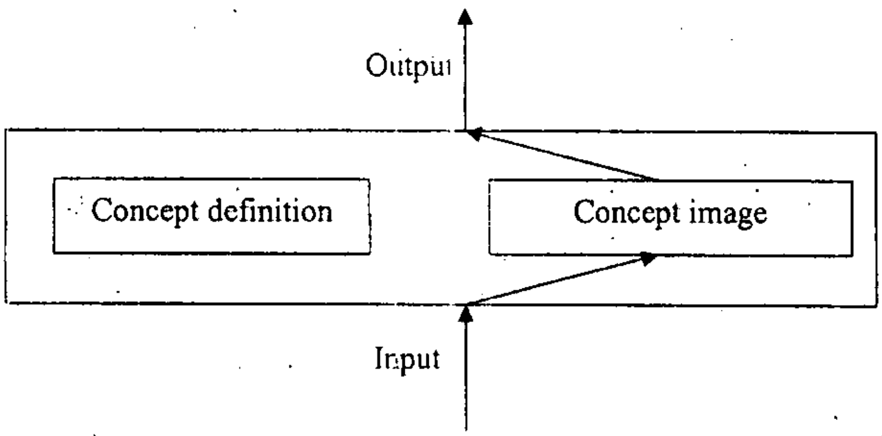


Figure 5.4 Intuitive responses

In the first three processes when a task is given to a student s/he consults the concept definition before the answer is given It does not matter if the concept image is consulted before or after the concept definition or if it is not consulted at all. In the last process the student responds to the question by consulting the concept image and without consulting the concept definition. Vinner indicated that it is this last process that is most commonly used by students. In this study, the way the students answered the questions, in which the three representations were related, they often ignored their concept definition or even contradicted it. Table 5.1 shows the relationship between the concept definitions and concept images for the students who were interviewed.

Table 5.1 Students' concept images and concept definitions

Name	Concept definition	Concept images		
		Numerical representation (Q 4.2 and Q 6.4)	Graphical representation (Q 6.3)	Algebraic representation (Q 6.7 and Q 7.3)
Adino	Notation/ algebraic calculation	- motion - asymptotic	- asymptotic - barrier	- **procedural**
Almeida	Approach	- **motion** - asymptotic	- **motion**	- procedural
Artur	Notation/ algebraic calculation	- **value correspondence**	- asymptotic	- **procedural**
Bertino	Approach	- procedural - **motion**	- procedural - asymptotic	- procedural
Gomes	Approach	- procedural - **motion**	- barrier - **motion**	- procedural
Humberto	Boundary	- procedural - motion	- procedural - **asymptotic** - **barrier**	- procedural
João	Notation/ algebraic calculation	- **procedural** - motion - asymptotic	- motion	- **procedural**
Pedro	Boundary	- motion - procedural	- **asymptotic**	- procedural
Souto	FD	- procedural - value correspondence	- asymptotic	- procedural

In the table I have highlighted the students' concept images that matched the concept definitions they held. The table shows that the interviewees seemed to hold four concept definitions: the *formal definition*, a *notation/ algebraic calculation*, a *boundary*, and an *approach* concept definition. Only one student evoked the *formal definition* to reproduce the definition. However this concept definition did not match with the images he recalled to solve tasks within the three representations. All three students whose definitions were dynamic in nature, namely, an *approach* concept definition used a *motion* image when working in the numerical task, and two of them when faced with a task in the graphical representation. This revealed that the concept definition *approach* by and large fitted the students' concept image recalled when working in the numerical and graphical representations. To perform the graphical tasks two students, whose concept definition was *boundary*, used the *asymptotic* image. Their image used in answering questions in the graphical representation was commensurate with the concept definition they held, but there was an inconsistency between this concept definition and the images invoked in the numerical and algebraic representations. The *notation/ algebraic calculation* concept definition given by three students was reliable with a *procedural* image they used in the algebraic representation, but it was inconsistent with the images evoked to solve the task in the numerical and graphical representations. João's use of a *procedural* image in the numerical representation was an exception. To solve Question 4.2b) some students (32%) shifted from the numerical to the algebraic representation before indicating the limit (see Chapter 4.3.2.1). This indicated the dominance of the *procedural* image when solving tasks which was dissimilar from most of their concept definitions. The concept definitions offered by those students were *approach*, *boundary* and *notation/algebraic calculation*. Similarly, 18,6% of the students resorted to formulas such as $y = b^x$ and $y = \dfrac{\log x}{x-3}$ to help them complete a table of values of a given graph (Chapter 4.3.2.1). In some of the cases, the *procedural* image they evoked to solve the task was inconsistent with the concept definitions they held. Rita and Arnaldo, who used the above analytic expressions, gave a *continuity* and *boundary* concept definition respectively.

I will use examples from student work to illustrate the relationships in Table 5.1. Firstly, students were asked to represent $\lim_{x\to 2} f(x) = 5$ graphically. Pedro's definition of a limit was, "*limit of a function is the maximum value, where the function ends*". The words 'maximum' and 'ends' evidenced the image as a boundary. Therefore his definition was classified as static, more exactly, as a *boundary* concept definition. This boundary might or might not be reached but is not surpassed. Also Pedro' s graphical representation for Question 6.7(b) is given in Figure 5.5.

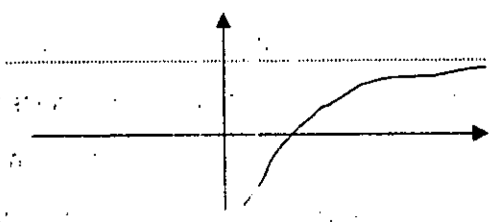


Figure 5.5 Pedro's graph of $\lim_{x\to 2} f(x) = 5$

The graph presented a horizontal asymptote and the curve continually approached this straight line, without touching it. It appears he evoked an *asymptotic* concept image to operate in the graphical representation. Accordingly, there is a relationship between his concept definition, a boundary that was never touched, and the *asymptotic* concept image.

In a second example, Gomes defined a limit of a function as "*the value to which the function tends when x tends to a, in other words is the behaviour of the function when x tends to a real number*". Words such as "tends to" and "behaviour of a function" inferred a dynamic character within his concept definition. This concept definition was classified as *approach*, meaning that the sequence of y values gets closer to b when the corresponding sequence of x values approaches a. In the interview, when faced with the numerical question, he looked at the behaviour of the function

> "I calculated, or saw the limit principle, all y values increased according to the x values assumed. So here, after the value 1,99 the next point is 2,01. So I used the same behaviour for the y values"

This protocol illustrates that Gomes' concept image also had a dynamic character, one classified as *motion picture*. Again, there was commonality between the concept image and the concept definition. Although the concept definition did not contradict the concept image we cannot say that the students consulted the concept definition or concept image before they performed the task. If they did, their actions certainly did not reflect most of the students' actions. Most of the students appeared to develop a concept image that was not consistent with their concept definition. The concept definition appears to be ignored when the students worked in most of the various representations. An illustration thereof in the study was Souto and João's work. Souto attempted to give a formal definition as follows,

> "limit of a function is understood as a point $x = a$ in the interval of the domain of the function, so that exists a number $\varepsilon \rangle 0$ it is $\varepsilon \rangle a$ where $f(x) \rangle f(a)$, so it means limit of a function when x gets to a is equal to A, i. e. $\lim_{x \to a} f(x) = A$".

However, the sentence did not match the meaning of his formal definition. He might have rote-learned the definition and when he was required to reproduce the definition he distorted it. When he worked in the numerical representation he seemed to use a *procedural* concept image. Actually, he resorted to an analytic expression to evaluate the limit instead of analysing the neighborhood of the given point according to his concept definition. Similarly, he used the *procedural* image when working in the algebraic representation. Alternatively, in the graphical task he used the *asymptotic* image. Consequently, it appears that there was no association between this students' concept definition and concept image.

Furthermore, in his definition, João claimed that the "limit of a function is $\lim_{x \to a} f(x) = A; f(x) \neq 0$", which was classified as *notation or algebraic calculation*. In the interview, he was asked to explain how he sketched $\lim_{x \to 2} f(x) = 5$. He answered,

> "Here, when the values of x approach 2 the function is getting closer and closer to 5. That is why I drew the graph so"

This statement illustrated the student's perception of the dynamic character of the limit concept. This concept image was one of *motion* and was inconsistent with his concept definition. Conversely, a *procedural* image evoked when he solved algebraic tasks was coherent with his concept definition. It appears that the last two students did not recall their concept definitions before carrying a task. If they had consulted their concept definitions they would have exposed a conflict between the concept definitions and concept images evoked in the solutions process. For instance, in Souto' s *formal definition* he elaborated the intervals and neighborhood of a value a, which had little or nothing in common his perception of a movement of the points of a function towards a value evident in his concept Image.

The research findings corroborated Tall's (1991) view that the students, when performing a task, most often do not consult their concept definition. It appears that they solved the task independently of their concept definitions, but with the use of their concept images. For example, some of them used a *procedural* image, sometimes inconsistent with their concept definitions, to solve any task, because their concept image of a limit of a function was *a* sequence of calculations employing rules and methods.

5.1.2 The students' concept images and the representations

In this Section I want to summarize the concept image derived from the students' work in the different representations Doing so, I will bring together the two fundamental theoretical frameworks on which my study was based, the concept definition/image (Tall and Vinner, 1981) and the representation theory of Douady (1986). The mental representations forming the concept images are materialized through physical representations systems such as the graphical, numerical and algebraic representations. Douady (1986) claimed that the use of different representations enhanced the students' conceptual development. Thus, I analysed both, representations and concept images. As was mentioned in Chapter 4.2, the students held multiple concept images either in the various representations or within the same representation, but as Table 5.1 portrayed the *procedural* was the dominant concept image.

The relationship between representations and concept images had two facets, that is, the concept images and representation systems mutually influenced each other. On the one hand, the students' concept images influenced the choice of the representation they used to solve the task in some cases. On the other hand, the representation in which the task was given led the students to recall a suitable concept image. Firstly, it appears that for the majority of the students, the concept images they held drove them to use a specific representation. For instance, as we saw in Chapter 4.3.2.1 and 4.3.2.2 the students resorted to an analytic expression (algebraic representation) to indicate a limit or to complete a table given a numerical table or a graph respectively. They conceived of a limit of a function as a *procedural* concept image, therefore they performed the tasks algebraically irrespective of representation. In addition, in the numerical task some students found an analytic expression that did not match the given pairs of values of the table. Consequently, the limit they obtained did not go with the other values of the table. However, the students did not reject these solutions. This demonstrated once more, the dominance of the *procedural* image that guided the students to an algebraic representation and their trust in algebraic solutions. Conversely there is evidence that the representations used in some questions guided the students to use different concept images. For example, for some students the limit of the function represented in Figure 5.2 (graphical representation) was equal to 3 (Question 6.3(d)). The graphical representation guided them to recall a *motion* concept image, where the values of the function approached and reached the value 3. However the algebraic representation $\lim_{x \to +\infty} f(x) = 3$ (Question 6.7(a)) led them to an *asymptotic* concept image. They sketched the function with a horizontal asymptote $y = 3$.

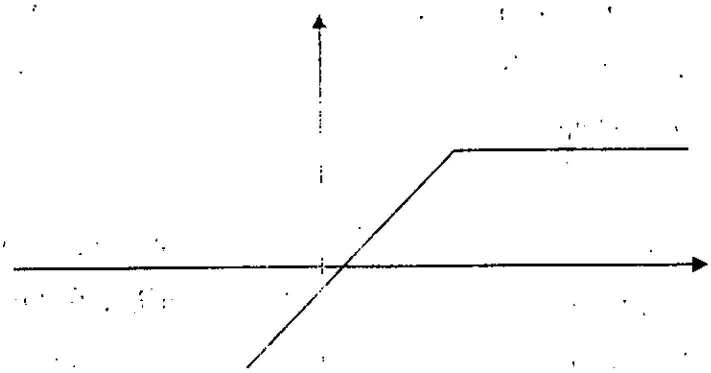


Figure 5. 6 What is the limit of the represented function?

It appears that previous knowledge about a constant function dominated the student' s choice of the concept image where the y values approached and reached the value 3, whereas the symbol $+\infty$ might stimulate the image of a horizontal asymptote. A further example was, Martina. In Question 3.3 she claimed that, "$\lim_{x \to 4} f(x) = 1$ *means that when x approaches 4 the function will have its maximum value 4*". In this case she evoked a *barrier* concept definition, but she also used an *asymptotic* image to sketch the graph of $\lim_{x \to 2} f(x) = 5$ (Question 6.7(b)). In addition she applied a *procedural* image to solve Question 4.2 given in the numerical representation. It might be that the symbol 'lim' (limit), used in Question 3.3, reminded her of a boundary, thereby evoking a *barrier* image. Alternately, in a similar question (Question 6.7(b)) she might have associated the limit with the verbs 'tends to' or 'approaches to' that suggested a continual movement to an asymptote. And finally, in Question 4.2, the *procedural* image appears to have dominated all other possible concept images associated with the numerical representation. Similarly, Higuera et. al. (1994) reported that the students used proprieties of functions (continuity and increasing), in a graphical representation and not in an algebraic representation, to justify the existence of a function. The reason Higuera et. al. (1994) gave was that students considered the proprieties as graphic characteristics and not analytic ones. The example shows that only the graphical representations stimulated the continuity and the increasing concept images This agrees with the influence of the representation form in the students' choice of the concept image and consequently how they solved the task. In the next paragraph I put together Tall's concept image and representation theory of Douady using students' work.

Douady (1986) suggested that developing a concept using different representations help students learn new features that are more visible in a particular representation. Indeed, as the student proceeds from one to another representation s/he expands his/her cognitive structure with aspects s/he did not notice within other representations, and forms a richer concept image of the considered mathematical object. This happens when the images s/he forms working with each representation are in harmony with the images s/he forms within other representations. Unfortunately, in most of the cases students developed different and sometimes contradictory images when working in different representations, and they do not notice the contradiction if they did not evoke these images simultaneously. I picked two examples from my research to illustrate the connection between concept image and representation theory. Artur's definition of the limit concept was classified as notation/algebraic calculation (Table 5.1). When he worked within the numerical representation he used a value correspondence image, where the limit is a value of the function. However, he used a procedural image within the algebraic representation and chose an asymptotic image to solve a task within the graphical representation. Thus, he drew a curve approaching a straight line without touching it. This image is not consistent with the image he used within a numerical representation. In addition, he was not able to graph the limit result of Question 7.1c) in the task, and in the interview he stated that it is not possible to draw a graph with that information, he needed other information such as zeros of the function, the y-intercept(s) and so on. This suggested that he did not develop an asymptotic image within the algebraic representation. Conversely, Bertino used a uniform concept image (procedural), not consistent with his concept definition (approach), for solving tasks within graphical, algebraic and numerical representations (Table 5.1). The examples above showed that Arthur used different and contradictory concept images depending on which representation he was working on. Artur's work evidenced that the representations used had an effect on the concept images he evoked to solve the tasks. On the other hand, Bertino recalling a procedural concept image resorted to an algebraic expression irrespective of representation (numerical and graphical). Thus, according to the concept image he evoked, he worked algebraically in all tasks no matter in which representation the task was given.

In conclusion there is a dual (non-causal) relationship between the representation and the concept images chosen by students to solve a task. Secondly, the students held multiple concept images of the concept limit of a function. When they faced a task, dependent on the given representation they recalled a suitable concept image. This does not mean that there is one dominant image in each representation. As Table 5.1 shows some students used more than one concept image in the same representation. Thirdly, the research findings demonstrate that the representation used in a question also stimulated the students' choice of a specific concept image to perform the task.

5.2 SUMMARY

5.2.1 Purpose of the study

With this study I intended to ascertain Mozambican students' concept definitions and concept images and how they worked within systems of representing limits. The answers to the three research questions helped me to draw my conclusions. The research questions were the following:

1. Do the students conceive of the limit as a static number or as a dynamic process?
2. What concept image and concept definition of the limit of a function do the Mozambican Grade 12 students hold?
3. How do the Mozambican Grade 12 students understand each representation and how do they relate the different representations: theoretical (definition), graphical, numerical and algebraic?

5.2.2 Primary findings of the study

The present study was undertaken in two classrooms from two Mozambican schools, one in the capital, Maputo, and other in Quelimane, a northern province. Firstly, nineteen students (26,8%) left Question 2, about the definition or explanation of the limit of a function, blank. The answers to this question offered the different concept definitions held by the students. From the nature of the students' definitions in Question 2 they held either a dynamic (16,3%) or a static (56,9%) concept definition of the limit of a function (Table 4.1 in Chapter 4.2). The static concept definitions (notation/algebraic calculation, boundary and formal definition) suggested a fixed number or distance and included terms such as the last value, the maximum or the minimum value of the function, interval, neighborhood or a definition with $\varepsilon - \delta$

symbols. The majority of the students (62,5%) used a static concept definition, a notation or a sentence meaning an algebraic calculation reflecting their practice in the classroom. Moreover, nine students assigned a limit as a boundary in agreement with concept definition held by students in other studies (Davis and Tall, 1991, Monaghan, 1994). As revealed by the available literature the formal definition using the $\varepsilon - \delta$ symbols seemed to be too complicated for the students (Tall and Vinner, 1981, Vinner, 1991). Only one out of four students who tried to write a formal definition wrote a complete sentence, and it was not understandable. Another student offered an incorrectly worded formal definition. Alternatively, some students (16,9%) used a dynamic concept definition (*continuity* and *approach*) that incorporated an idea of movement towards a target, which might or might not be reached (Chapter 4.2.4 and 4.2.5). The *approach* definition echoed a widespread idea of a limit as a continual movement. Tall (1991) indicated that the words "approaches but not reach", "cannot pass" and "tends to" used in such definitions are associated with a process of "getting closer" or "going on forever" expressing a dynamic view of the limit concept.

Secondly, the mental images the students formed about a concept show how well they understood the concept. Tall and Vinner (1981) denominated these mental images *concept images*. The concept images generated from the students' answers to Questions 4.2, 6.4, 6.3, 6.7 and 7.3 were divided into dynamic (39,3%) and static images (60,7%) as summarized Table 4.3 in Chapter 4.3.1. The *motion picture* and *asymptotic* images gave the idea of movement towards a value, which was reached in the first case and not reached in the latter. The static images included a *barrier*, *value correspondence* and the *procedural* images. These images portrayed a limit as a fixed value that might not be a value of a function (barrier) or that was necessarily a value of a function (value correspondence) (Chapters 4.3.1.3 and 4.3.1.4). Moreover, the limit was portrayed as a value that resulted from successive mathematical calculations (procedural) (Chapter 4.3.1.5). This image predominated the students' work. As was mentioned in Chapter 4.2.1 this was a result of what they frequently faced with in the classroom.

Thirdly, the mental images come into being within an external representation (Dreyfus, 1991) and the switch between representation systems enhances students' evolution of a concept (Douady, 1986). Therefore, questions asking the students to move from one representation to another were developed in my research. The analysis of students' papers showed that they were more confident working in the algebraic representation, most possibly due to the time they had devoted to exercises of limit calculations at school (Chapter 4.2.1). They faced difficulties when switching from one representation to another and sometimes obtained contradictory responses. However in most cases they did not notice the contradiction. It seems that the students compartmentalized the representations. A phenomenon is compartmentalized when one has two different conflicting schemes in his or her cognitive structure (Vinner and Dreyfus, 1989). The students did not understand the relationship between the representations as was referred to in Chapter 4.3.2.

Fourthly, an analysis of students' concept definitions and concept images indicated that the majority of the students mostly conceived of the limit as a static number, without any relation with the limiting process. Only a small number of the students appeared to conceive of the limit as a dynamic process. This finding was not in accordance with previous results reported by various researchers. According to Monahagan (1994) various researchers found that the students conceive of the limit not as a static concept, but as a dynamic process.

Fifthly, the student's understanding of a concept may be measured by his (her) ability of modeling reality to solve mathematical problems. Mathematical expressions are often constructed to analyse objects or processes (Dreyfus, 1991). In the process of solving a task modeling reality students must distinguish concept images produced by the situation or the object modeled from those that are generated by the mathematical expression used to model reality. The students in the study revealed shortcomings when working with a mathematical expression that represented the decreasing of temperature of a body removed from the fire. They mismatched mathematics concept images and the contextual concept image and sometimes obtained solutions that contradicted the modeled reality.

5.3 IMPLICATIONS OF THE STUDY

The Mozambican students faced difficulties when they encountered the concept of a limit of a function. This may be because of the abstract nature of the concept. Various researchers have also reported similar difficulties faced by students in different countries (Cornu, 1991; Bezuidenhout, 1999, Robinet, 1983). With the findings of this study I intend to reach three goals. Firstly, research in education is still very new in Mozambique. A few people in Mozambique have been investigating the topic, a limit of a function. Therefore, one of my goals is to inspire other Mozambican investigators to work on this basic concept in the Calculus. Secondly, I hope to call Mozambican policy makers attention to the necessity of awareness of the elaboration of curricula, manuals and in the teachers' training programs so that new teaching methods may be introduced. Thirdly, this study is mainly directed at the teachers. I hope that the findings of this study help them to be more aware of some difficulties that the students face when encountering the concept of a limit of a function, and call their attention to the use of multiple representations, and the benefit of applying the limit concept in reality for the evolution of students' conceptualization. Fourthly, within the available literature I did not find a systematic investigation of the relationship between the concept image (Tall and Vinner, 1981) and the representation systems (Douady, 1986). I hope that I raised an issue that will be a contribution to future research.

5.4 LIMITATIONS OF THE STUDY

Since I worked only with two schools, this is a restriction to generalize the results. In addition, the task and the interviews were developed in Portuguese. I am not fully proficient in English and it is possible that some bias was introduced in the translation of the interviews transcripts into English and in the meaning of some words used to categorize the students' answers. To overcome this, I discussed the sense of the words with my supervisor and colleagues in Maputo.

5.5 FURTHER RESEARCH QUESTIONS

Firstly, only two classes undertook the study. So, I suggest a similar study applying a large number of students, so that a generalization can be drawn.

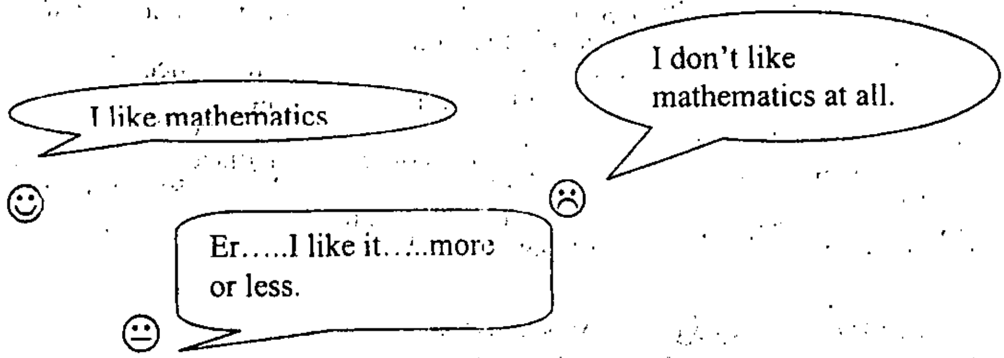
Secondly, the interplay between concept definitions and concept images seemed to be important for the students' actions when facing a task. In my study I analysed some cases where the concept images matched or did not match with the concept definitions. Of interest would be to go further and see if there are cases where both concept images and concept definitions are actually evoked and how it happens. This means, which of them is recalled before or after the other.

Thirdly, I suggest a deeper investigation into the relationship between the representations used to perform the task and the concept images held by the student. Specially, because I found only one example that evidenced that the concept image influenced the choice of the representation to solve a task. Namely, the *procedural* image, that appeared to dominate the students' way of solving tasks.

Annex A

Let's speak about mathematics

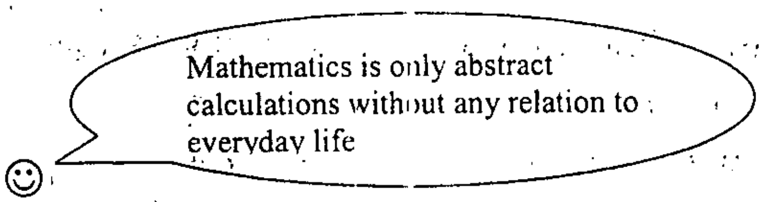
Question 1



1.1. Put a circle in the word that correspond to your feeling about mathematics.

very much	much	I like it	more or less	not at all

1.2. One student makes this claim:



What is your opinion? Put a circle in the appropriate choice.

strongly agree	agree	strongly disagree	disagree	don't know

Question 3

3.1. Write down a definition of the *limit of a function*. If you don't remember it try to explain with your own words what you understand about the *limit of a function.*

3.2. Some students states that the definition of limit of a function is difficult to understand. Others do not have problem with it.

☺ I understand the definition of the limit of a function very well.

☹ I don't understand the definition of limit of a function at all.

How well do you understand the definition of limit of a function? Encircle one of the boxes.

Very well	Well	More or less	Not at all

3.3.a) Write one sentence in a non-mathematical context using the phrase *tends to*

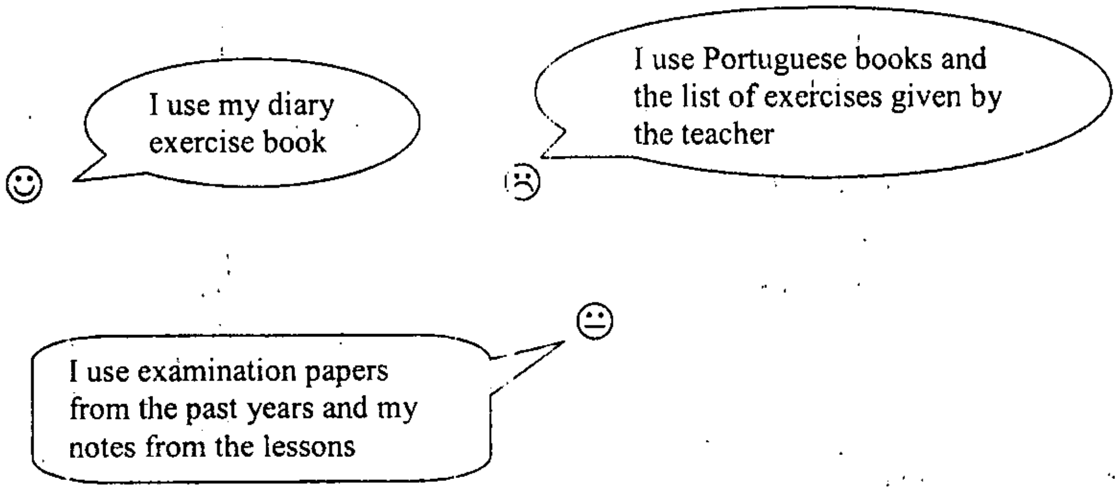
b) Write one sentence in a non-mathematical context using the word *approaches*

c) Write one sentence in a non-mathematical context using the word *converges*

d) Write one sentence in a non-mathematical context using the word limit

3.4. Explain what $\lim_{x \to 4} f(x) = 1$ means to you.

1.3. Which materials do you use to study mathematics? Put a circle in the appropriate choice. (Note you can choose more than one letter)



Diary exercise book	list of exercises given at school	examination papers from the past years	other materials

Question 2

Try to rank from 1(least time) to 4 (most time) the time you used in the lessons about *limits of functions* dealing with:

- theory (definition and theorems) ☐

- calculus ☐

- solving problems related to daily life ☐

- working with graphs ☐

Question 4

4.1. 1. Do you use a calculator during your mathematics lessons?

Yes	No

4.1.2. Which calculations do you normally do in the *limit of function* with the calculator?

4.2. Examine the table below, which represents points that belong to a linear function.

X	2	3,4	3,49	3,5	3,51	3.9	3,99	4	4,01	4,1	4,2
Y	-1	0,4	0,49	0,5	0,51	0.9	0,99		1,01	1,1	1,2

a) As x gets closer to 3,5 what happens to y?

b) What is the value of $\lim_{x \to 4} f(x)$?

4.3.a) Transpose the points to the graph

c) Show on the graph the value the function approaches when x gets closer to 4.

Question 5

5.1.1. Do you apply the concept of limit of a function in other courses in Grade 12?

Yes	No

5.1.2. Name these courses?

5.2. A metallic bar was removed from the fire. Its temperature, starting from that moment, is given by the expression:

$$T(t) = 26 + \frac{90}{t+3}$$

T is the temperature in degrees Celsius and

t is the time in hours.

a) **What was the temperature of the bar as it was removed from the fire?**

b) **What happens to T as t increases?**

c) **The bar is getting cooler until it reaches room temperature. What is the approximate temperature of that room?**

d) **Determine** $\lim_{t \to +\infty} T(t)$

e) **According to the problem what information give you the** $\lim_{t \to 6} T(t) = 36$

Question 6

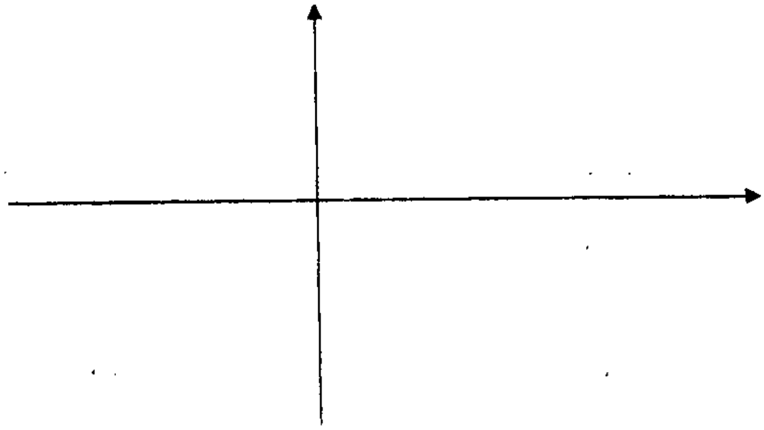
6.1. Given the function

$$y = \frac{1}{x-2}$$

a) **Draw the asymptotes of this function in the given Cartesian system.**

b) **Find the x/y intercepts**

c) **Using your answers to a) and b) draw a rough sketch of the graph**



d) What happens to y when x gets to 5?

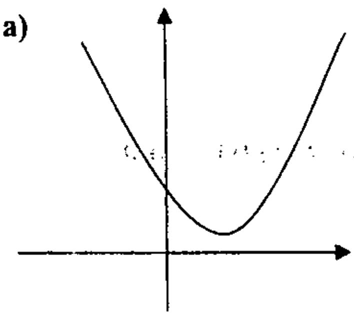
e) What happens to y as x gets closer to 2?

6.2. Determine the following limits

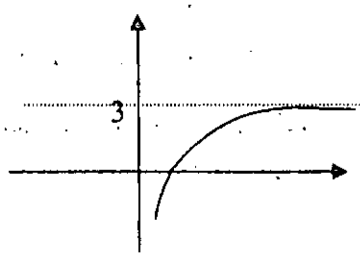
$\lim_{x \to 2^+} \frac{1}{x-2} =$ and $\lim_{x \to +\infty} \frac{1}{x-2} =$

6.3. What are the limits of the functions when x gets bigger in each case?

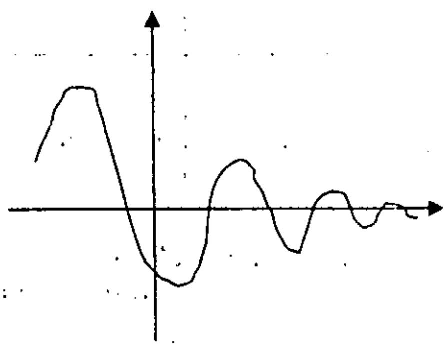
a)



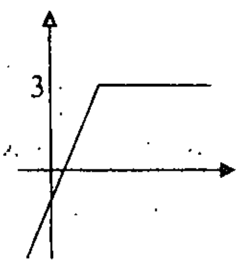
b)



c)



d)



6.3.Complete the table of values of the function in b) as best as possible

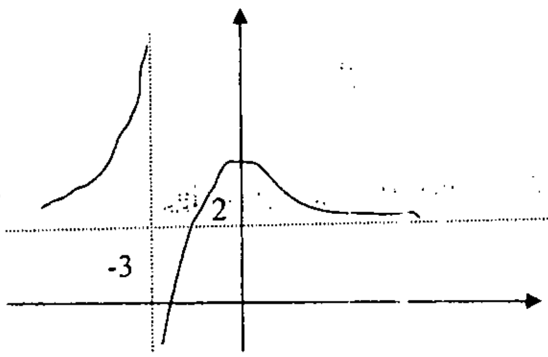
x	1	2	100	10000	100000000
f(x)					

6.4.What role the asymptotes play in the graph of a function?

6.5. What are the asymptotes of each of the following functions?

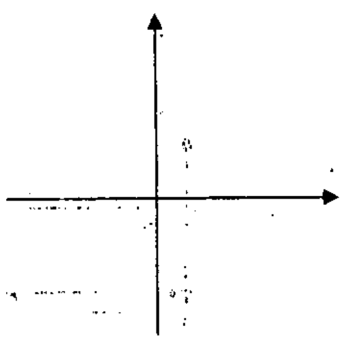
$a)\ y = \dfrac{x^2 - 4}{x - 2}$ $\qquad$ $b)\ y = \dfrac{x - 5}{x - 2}$

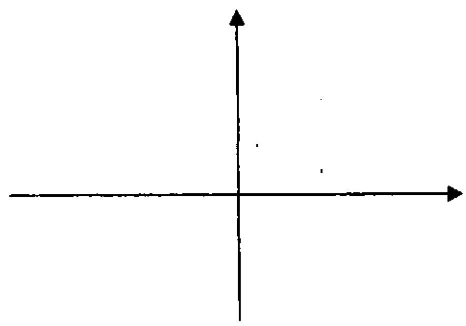
c)



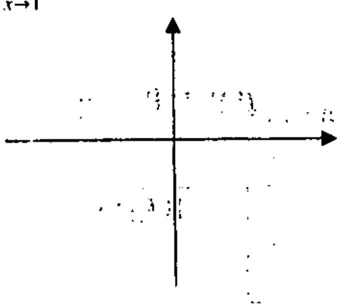
6.7. Draw a rough sketch of a possible graph to show the given limits.

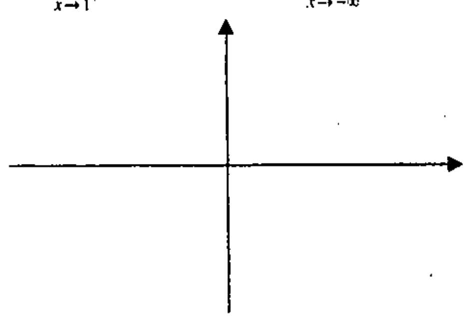
$a)\ \lim_{x \to +\infty} f(x) = 3$ $\qquad$ $b)\ \lim_{x \to 2} f(x) = 5$





$c)\ \lim_{x \to 1^+} f(x) = -\infty$ $\qquad$ $d)\ \lim_{x \to +\infty} f(x) = +\infty, \lim_{x \to 1^+} f(x) = 2$ **and** $\lim_{x \to -\infty} f(x) = 3$





Question 7

7.1. Solve the following limits

a) $\lim_{x \to +\infty} (x^2 - 3x + 1) =$

b) $\lim_{x \to 2} \frac{2 - x}{x - 3} =$

c) $\lim_{x \to 3} \frac{x + 2}{x - 3} =$

d) $\lim_{x \to 2} \frac{x^2 - 5x + 6}{x^2 - 4} =$

e) $\lim_{x \to 9} \frac{x - 9}{\sqrt{x} - 3} =$

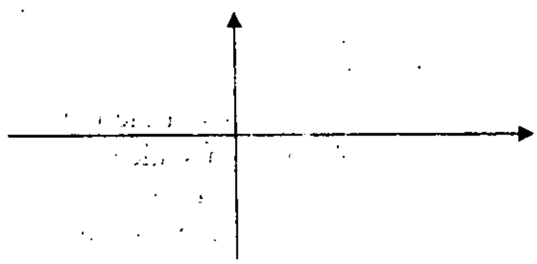
f) $\lim_{x \to +\infty} (1 + \frac{1}{x})^{3x} =$

g) $\lim_{x \to +\infty} (\frac{x + 3}{2 + x})^{x} =$

7.2. Use c) to complete the table

x	2,4	2,8	3	3,2	3,4
y					

7.3. Draw a rough graph to illustrate your answer to 7.1 c)



7.3.1. Which one of a-g in 7.1 seemed more difficult to you?

7.3.2. Why?

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